An Approach to Number Education in the Infant School

COUNTING AND MEASURING

by EILEEN M. CHURCHILL

Recent research findings suggest that the basic principle and concept on which all mathematical thinking rests can be developed in the early years of childhood and in this book Miss Churchill examines the nature of these concepts, and from a wealth of experience with children suggests the kind of opportunities which can contribute to their development. The goal of education in numbers is conceived as the establishment of certain ways of thinking about and ordering experience which are connected with the nature of the number itself, and indeed of all mathematical symbolism.

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AN APPROACH TO NUMBER EDUCATION IN THE INFANT SCHOOL

by

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FOREWORD

THE key to Eileen Churchill's treatment of the beginnings of mathematical understanding in childhood is to be found in the following quotation:

> The story of number . . . covers a long period in the history of mankind. This is something the educator should remember. Today the little child hears the language of number and picks up many of the words as early as any others. This does not mean that he endows them with a mathematical meaning, and he must be allowed time to feel his way into the language as did his forebears.

It is all too easy to assume that teachers who have themselves been through the mill of arithmetic at school, and a certain training in method at college, can be relied upon to do all that is necessary to introduce children to arithmetic in the primary school—especially with the wealth of bright number materials now available. The principle that knowledge of number must grow out of concrete operational experience is well understood. But number is one thing; the numeral system is quite another. Familiarity with number combinations and operations can develop from manual and visual experience, but this of itself will take the child little beyond the level of palaeolithic man. Our numeral system, derived from the Hindus and Arabs, the product of centuries of mathematical exploration and invention, possesses such deceptive simplicity that it can be mechanically mastered with no reference to its history. And therein lies the danger.

The danger is not merely for the dull child. Indeed the danger is less for him because his capacity for insight is limited and mechanical arithmetic is the best he can hope to achieve. It is with the brighter children, and precisely because they are capable of insight and are less disposed to accept arbitrariness unquestioningly, that the deadliness of uniformed teaching is so immediate. Our decimal system, our rules of calculation, our arithmetical terminology all seem so natural and inevitable to those who have successfully mastered them by mechanical methods that to see it for what it is, a blend of logic, history and convention, seems not only difficult but unnecessary, and to show it as such to children almost impossible. Yet it is not only possible but essential if the foundations of mathematical understanding are to be well and truly laid.

An effective power to calculate depends on four things: number experience, number notation, number rules and a store of number facts. The ability to use this power to solve problems depends on a fifth requirement, viz. a grasp of mathematical language. This is something which goes beyond an understanding of arithmetical notation (which enables numbers to be translated into signs and vice versa). Mathematical language enables concrete numerical problem situations to be translated into symbolic relations amenable to rules of calculation. That mathematics is a language is now widely accepted but the implications of this have not been adequately expressed in educational terms. Facility in the translation process must be developed from the beginning, when the problems are inevitably simple, so that the child grows up, so to speak, bilingual, capable of easy transition between number-facts and symbolic forms. The foundations of algebra are then laid at the same time as the foundations of arithmetic, and this is appropriate, for arithmetic is one special kind of algebra. As Miss Churchill herself says:

> Unless the language of number is communicated to the child in such a way that it forces him to think relationally about events in his experience then it will not serve as a firm basis for later, more advanced mathematical thinking.

FOREWORD

Teaching the foundations of arithmetic demands insight into number, insight into language and insight into the learning process. These insights are not easy to come by, and unless method in college is related to philosophy, psychology and linguistics, arithmetical teaching will continue to be uninformed. Miss Churchill, out of the wealth of her experience, her extensive reading and the insight gained from her own research at Leeds, has brought together, in a concise and attractive form, the concepts of many thinkers not commonly associated in educational texts. Piaget is well enough known but Sapir, Whorf and Cassirer have not been shown hitherto as having a contribution to make to mathematical education. For the more inquiring students and teachers these hints of a fascinating world of thought may well open fresh horizons.

Although the main emphasis is on number, the problem of learning about space is broached. Those who are familiar with Eileen Churchill's assiduity in research will anticipate, from her current concentration on spatial concepts, that the present volume may well be followed by one on space having the same fertility, attractiveness and suggestiveness as the present volume and more besides.

G. PATRICK MEREDITH

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PREFACE

My primary purpose in writing this book has been to attempt an exploration of the aims and content of what might be called Number Education in the Infant School. To do this adequately, it seemed to me necessary to examine the nature of number concepts and the language of number as well as to consider the evidence available regarding the modes of learning open to young children. Chapters 1–6 are concerned with these considerations and provide the basis for suggestions regarding Number Education which will be found in chapters 7–10. Those teachers who are less interested in theoretical than practical considerations may prefer to read these latter chapters first, though it seems to me that both kinds of thinking are necessary if teachers are to provide intelligently for children's needs in this field of experience. We need to know not only what kind of experience to give them but what kind of learning we should be looking for.

It is not possible to mention by name all the many teachers, students and children who have contributed to the ideas expressed in this book, though I wish to acknowledge my debt to students at Leeds Training College who told me many of the incidents recorded here; and to teachers in Leeds, the West Riding of Yorkshire, and the City and County of Leicester who have allowed me to watch children and teachers at work together in their schools. I wish also to thank the authorities in these areas for permission to use this material, and in particular the Directors of Education of the City and County of Leicester for permission to reproduce photographs taken in their schools.

I owe particular thanks to Mr. Schiller, until recently Staff

Inspector of Primary Schools; to Miss Thyra Smith, retired H.M.I.; to Professor G. P. Meredith of Leeds University; and to Dr. Mowat, principal of Doncaster Training College. To these people I am indebted for much that I have learned and tried to express in this book, and I am specially grateful to Dr. Mowat for the helpful criticisms she made while it was being written.

Mr. Roadley of Leicester University Dept. of Education took the photographs reproduced in this book and I wish to thank him for all the trouble he has taken with them.

E.M.C.

Leicester,
September, 1960.

CHAPTER ONE

ABOUT CAROLINE AND OTHERS

Caroline is eight years old and nearing the end of her first year in the Junior School. I was visiting her class recently when the following incident occurred. Caroline was faced with a problem card which read, 'If I buy 165 tulip and daffodil bulbs and 49 of them are tulips, how many are daffodils?' Having read the question she took the card up to the teacher and asked, 'Is it a taking-away sum?' She received the reply, 'Go and have a try' and returned to her desk, whither I followed after a moment or two, curious to see what she would do. First of all she wrote down 49 and under it 165 thus:

$$\begin{array}{r} 49 \\ 165 \\ \hline \\ \hline \end{array}$$

She looked at this and then put a nought beside the nine and a minus sign alongside the nought. She then proceeded to subtract, arriving at 325. This she looked at for a moment and then, turning to me, she said, 'That can't be right, it's more than a hundred and sixty-five.' 'No,' I replied, 'can you see where you've gone wrong?' She couldn't, and by this time I was sitting beside her and suggested that we tried first with a smaller number of bulbs. 'If you bought twenty-four bulbs and six of them were tulips, how many daffodils would you have?' I asked. 'Eighteen,' came back the answer, without hesitation. 'Now can you write down what you have done in your head?'

I asked, and this is what she did. First of all she wrote down 6 and 24 underneath thus:

$$\begin{array}{r} 6 \\ 24 \\ \hline \\ \hline \end{array}$$

then she put a nought above the 2 and a minus sign alongside the 6. Her sum when worked out read as follows:

$$\begin{array}{r} 06- \\ 24 \\ \hline 22 \\ \hline \end{array}$$

and on looking at it she said, with some exasperation, 'But it ought to be eighteen, oughtn't it?'

This story may remind some of my readers of the story Miss Adams tells in her book[1] about Rosie and Billy. There seems to me, however, to be a significant difference between the two stories. As far as one can tell Rosie and Billy were quite content with answers giving values so remote from the correct ones that if they had had any real appreciation of the problems they were trying to solve, they could not have rested content with the results they arrived at. Caroline, on the other hand, can examine her answer and ask whether it feels right or wrong. This is, I believe, because she is now in a class where for nearly a year she has been having all kinds of experiences in which in order to solve problems, she has had to think about the relations involved in what she is doing; experiences of measuring, scoring, shopping, etc., in which the differences between quantities have been recognized by her as being important, experiences through which she is learning to estimate and evaluate, to compare and to share. During these activities the answers she records in terms of numbers express values which she understands, and because of this when she is faced with a problem at a more abstract level she is able to look at her answer and consider whether it looks right, something one cannot imagine Rosie doing in any circumstances, but certainly not while Billy remains her authority!

---

[1] Adams, L. D., *A Background to Primary School Mathematics.*

But Caroline is, at the moment, facing a particular difficulty due to the fact that she came up from the Infant School knowing herself to be 'good at sums'. She had in fact learned how to do quite difficult Arithmetical calculations. She could do additions and subtractions with hundreds, tens and units; multiplications with double figures, also division. Caroline had in fact filled a considerable number of books with sums of this sort, received a gratifying number of large red ticks, and was justifiably proud of her achievements, as were her class-teacher and her parents. But even now, after a year in her present class, her great desire when she is doing problem cards is to get the sum down, after which she proceeds audibly to repeat the pattern she memorized more than a year ago. As soon as she sees a sum written down she 'knows how to do it' and is experiencing a good deal of frustration when it lets her down.

My own interpretation of this story is that Caroline was taught to do sums with figures long before she was capable of understanding the abstractions with which she was dealing. Nor at this time was she being given the experience on which these abstractions are based. Hence her only method of learning was by committing to memory the rules of the game and practising them until she became an expert. Many children can be taught to develop this kind of skill while they are in the Infant School, and can achieve considerable satisfaction from 'doing sums'. It is only when one begins to watch the way in which they handle situations which, while requiring a familiar method of calculation, are nevertheless even slightly different from usual in the way in which they are presented, that one realizes how limited their skill really is.

Many children of Caroline's age are able to arrive at the correct solutions to quite difficult problems which involve numbers, but are unable to write down the processes by which they arrive at them, which raises several questions regarding the timing of instruction in written calculation. It may be noted here, in passing, that Caroline does not understand the function of the nought in our number system, nor does she understand the meaning of the minus sign. Both these points will be referred to later.

Caroline's method of dealing with sums is unfortunately not uncommon. Consider for a moment the following two examples

$$\text{(a)}\quad \begin{array}{r} 7\overline{)2345} \\ \underline{2} \end{array} \qquad \text{(b)}\ \tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}=\tfrac{3}{9}=\tfrac{1}{3}$$

A little patient analysis will reveal the reasoning behind both examples, yet one feels that Caroline would have been content with neither answer. The first comes from the book of a nine-year-old boy, but it may come as something of a shock to readers to be told that the second occurred in more than one examination script sent in by a group of training college students. It requires little effort to imagine what went on in their minds when faced with $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=$: an effort to remember how they did this kind of sum when at school, a lapse in memory regarding L.C.M.'s, a sudden recall of cancelling, and that's that one done!

When I was at school I was taught arithmetic, and later algebra and geometry, by method of rote-learning. A process was demonstrated on the blackboard and we wrote it down. We were then required to repeat it through a series of examples until a sufficient number in the class were gaining high enough marks to suggest to our teacher that we were ready for the next process. Through this method I learned to perform the tricks of the trade sufficiently well to pass my school-leaving examinations and it was only some years later when I was trying to apply my mathematical skill in experimental science that I began to realize how very little I understood, and how very little meaning the equations I arrived at had for me. I was able, more often than not, to arrive at the 'right' answer but did not know how to interpret it when I had got it. Moreover, I discovered that I only knew which of my tricks to apply if I was shown the method. I was often in a similar position to Caroline though my question was not, 'Is it add up? take away? or multiply?' but more often something to do with 'Which method of correlation shall I apply?' and frequently I must admit to having been as much at sea as they were. At first in this situation I came to the conclusion that mathematics would always be a closed book to me, but as I made the effort to understand I found it far easier than I had ever dreamed possible, and then I began to be very critical of the teaching I had received at school. I realized that had the appeal been to my reason and imagination instead of to my memory I should almost certainly have been able while at school to take a delight in mathematics and to appreciate its uses and powers.

As I have been concerned for half my professional life with the training of teachers I have naturally given a good deal of thought to this particular aspect of education and have become convinced that the mistakes begin in the Infant School. The Training Colleges are justified in asking the Grammar Schools what they have been doing for the previous six or seven years when students arrive with as little understanding of the simplest arithmetical processes as some of them do, but equally Grammar Schools may ask the same question of Junior Schools particularly as they are supposed to receive only the so-called 'cream'. But it seems to me that the Junior Schools would find their task easier were children to arrive with some understanding of the basic number concepts rather than with a number of skills with numbers mechanically performed.

This is one side of the picture but there are other children I want to talk about. Only yesterday my four-year-old niece decided to decorate her aunt's head with feathers. When she had finished she said to me, 'You've got three at the back and one at the front.' Clearly Peggy knows something about three and one and the difference between them. But another aunt I know told me this story. One day she took Catherine and Gillian, who were both four years old, for a walk in the park.

After a time they sat down on a seat with auntie in the middle and presently Catherine leaned across her auntie and said, 'I'll count us—One, two, three; that's wrong I'm four; I'll try again,—One, two, three; but I'm still four. You try.'

This story illustrates one of the difficulties about numbers. They can be used in so many different kinds of situation which appear to have nothing in common. Clearly, the concept of four as expressing the number of years you have lived is much more difficult to understand than three as expressing a collection which you can see and handle.

Birthdays are important landmarks in one's life, however, and 'I'm four' says something very important about me. Recently Stephen arrived at the very important age of five while he was attending a nursery school where the cook always made a special cake to celebrate these occasions. All the children were looking forward to its arrival with excitement but there was a storm of protest when it did arrive with only four candles on the top. 'But Stephen's five,' they shouted, and another candle had to be fetched and lighted before tea could continue.

Another Stephen who was five years old had painted a picture of a lake with some ducks swimming one behind the other. The teacher paused beside him to admire the picture and then said, 'I see you've got five ducks on your pond.' Stephen looked at his picture in a rather puzzled way and then said, 'It isn't five, there isn't one in the middle.' All Infant teachers will know the source of Stephen's concept of five, and this story may cause them to reflect whether in fact we do not sometimes make it difficult for children to make the generalization from experience which turns the concrete number into the abstraction which he can use whenever appropriate.

The fact is that children pick up the number names, as they do other words, in particular contexts, and they are at first tied to these situations. Two may be a word you say when you see your shoes or put on your gloves, but you have taken a big step forwards when you understand *why* you say *two* about your shoes, when you understand the twoness of two and can recognize it when you come across it in other situations. Similarly, one, two, three, four, five may be words you learn to say when you are being helped up the stairs to bed by Mummy, but it may be a long time before you understand why you can also say them when you are threading beads on to a string.

The fact that we use numbers cardinally to express the sizes of collections, and ordinally to define position in a series are two concepts which have to come together for a full understanding of numbers, but they look like two completely different activities at first and remain distinct for quite a long time as Piaget[1] has shown in a work which will be discussed later.

I think all Infant teachers would agree that most children come to school with a good many number words already in their vocabulary and with the ability to use some, though not by any means all, in situations where they apply. We have, I think, in the past been misled by this vocabulary because we have not been sufficiently observant as to how and when it is used. Because of this we have assumed a readiness for sums which in fact is not present except in very few exceptionally bright children. Our task in the Infant School is rather to help children to get ready for this stage by giving them experiences through which they can develop the basic concepts necessary to an understanding of number. This to me is the primary purpose of number education during the Nursery and Infant years. Later in this book we shall explore the implications of this statement more fully. Meanwhile the following stories will indicate the kind of experiences which have been educative for certain children I have known.

Jacqueline, at four years old, was in the habit of counting everything she could find to count: dogs, trees, houses, etc. She discovered that the number marked on her gate did not correspond to the number she arrived at when she counted in ones from the beginning of the street. Her mother explained to her that the houses on one side of the street had been given the even numbers and those on the other side the odd ones. After this every time she went out Jacqueline insisted on counting the houses on both sides of the street, one lot as she went out, the others on the return journey.

---

[1] Piaget, J., *The Child's Conception of Number*.

Some four-year-olds were threading beads when David called out to the teacher, 'Look, my thread is longer than Jean's 'cos I've got eight on mine and she's only got three.' Kenneth, aged five, was playing a home game using a dice. He took much longer to shake the dice than the other children and after a very long shake, he announced, 'I'm shaking hard because I want a six.' Five-year-old Harry was helping to give out the milk. He approached the teacher and said, 'There are only five bottles left and six children without any. Shall I go and ask Miss Manning for another one?' Brian was playing a game of skittles with another six-year-old. He knocked some down and without counting turned to his companion and said, 'There are seven skittles knocked down. I know that because there are three left standing up.' John, aged seven, was asked to put out the mats for P.E. in four piles. He asked how many were needed and when told forty-eight he said, 'Four times twelve is forty-eight, so you'll want twelve in each corner.'

I have beside me as I write a book of sums written by Linda in the last two weeks of her Infant School life. The book is full of pages in which she has made up what she calls 'doubling sums', others where she has added in tens, and still others where she has halved. This is not the first book Linda has filled this year. They are done in time when she may choose to do what she likes and this, at the moment, is her usual preference. Numbers and what you can do with them fascinate her and she is not the only child in this class of whom this is true. One boy showed me a book in which he was making a record of the distance in miles between various places he had visited. He was using for reference an A.A. guide, a six-inch scale map of the area and a train time-table. The last-named was also being used by his friend who was finding out how long it would take to travel by train to various places the children were going to for their holidays, while yet another boy was writing a book about motor-cars which included all sorts of statistics which he knew much more than I.

These children have lived their school days, so far, in a school where no time is devoted to formal arithmetic teaching at all, and where any setting out of sums is only shown to the children when they ask for it in connection with some activity in which they are engaged. The activities are, however, carefully planned to involve the use of number which, as the headmistress expressed it to me, 'comes into nearly everything we do—you can't avoid it, can you?' It is this attitude, and the conscious exploitation of all sorts of situations which occur daily in the rich environment of a well-equipped Infant School, which have brought these children during the course of two or three years to the place where they now are. Judged by any standards their skill with numbers is at least satisfactory, but what is more important numbers have a meaning for them, they express relationships, are a method of comparison, and are exciting because of the patterns which you can find and make with them. Not all the children in the class have reached the level of those mentioned but a sufficiently high proportion for one to feel that the approach used in this school is justified. One feels that Caroline might not have been handicapped as she is at present had she had a similar experience while she was in the Infant School.

In attempting now to try and consider what are the aims of early number education, we have, I think, to pursue two lines of inquiry. We have to become clear as to the nature of number as well as the nature of children's learning at this age. It is as important to understand fully ourselves what we want them to learn, the ideas they need to acquire, the concepts which we shall be watching for; as to understand their modes of learning at this stage in their development and what they cannot possibly understand as well as what they can, given suitable experience.

Since some at least of my readers are likely to begin with the same handicap as myself the following two chapters are devoted to a study of the nature of number through an inquiry into man's use of numbering, beginning from the earliest times of which we have any record. This account is not included because it is thought that children in their development recapitulate the history of the race, though I think that little children have certain needs which primitive man also had and that these common needs give rise to certain interesting parallels in development which will be noticed from time to time as the tale unfolds. Primarily, it is included, however, because I found that my own research in this field, undertaken for another purpose, gave me an added understanding of the nature of number which encourages the idea that it may be equally illuminating to others.

CHAPTER TWO

THE BEGINNINGS OF NUMBERING

EARLY man was a hunter with few possessions and no settled community life. He survived if he learned to accommodate himself to his environment, to use what it offered in the way of food and shelter, and to distinguish the safe from the dangerous. It is not difficult to imagine how highly sensitive he would become to changes in his environment, and that it would not be long before he recognized certain sequences and patterns in natural phenomena. Any experience he gained which enabled him to predict the order in which things were likely to happen would no doubt add to his sense of security and ability to order his own life accordingly.

Space and time are the two fundamental realities with which he had to come to terms; awareness of patterns of relationships brings order into space and awareness of sequences produces temporal order. Life is full of patterns and sequences—night follows day and in turn there is another day. The moon rises as a thin crescent in the evening sky, on the next night it is broader, and this increase goes on steadily night after night till in time the full moon shines in all its glory. The stars are arranged in characteristic clusters and positions in the sky and the sun sets in different positions relative to them in a predetermined sequence, a sequence which takes longer to complete and would therefore be less readily apprehended than the moon-cycle. Birds and animals give birth to their young at certain fixed intervals, berries and fruits ripen at certain sea-sons. The tides also are rhythmical, and times of low and high tides may be predicted once the fundamental pattern is appreciated.

These rhythms in natural phenomena must have been of tremendous significance in the life of early man and one can imagine how in time he would come to appreciate that there was a certain predictability about the natural events which were so important to him. No doubt his first reactions to the sequence of day and night were of an organic order, a simple bodily response to light and darkness—he worked and ate while it was light and slept at night—a response as natural and physical as the sleep-waking rhythm of the new-born baby, and there is no reason to suppose that he thought about it, any more than does the baby. Nevertheless this basic organic response provides man with his first personal experience of temporal rhythm and from this develops in time the ability to perceive other rhythms to which he might with profit accommodate himself. No doubt the light of the moon would be of considerable importance and it would not be long before he came to expect the gradual waxing and waning, and by recording this to know how soon he might expect the next full moon. In time he would learn to relate this sequence to others in which he was vitally interested. At certain times there were ripe berries and fruits to be found which were good to eat, at other times there were none, but if he waited the berries would appear again. How long must he wait before he took a journey over the hill to collect them? How many times must the moon come to the full before he could expect another harvest of berries?

It is probable that in some such way the first impulse to reckon arose. The earliest numbering records we have are concerned with the marking of days as they passed, of their grouping into thirties according to the movements of the moon, culminating in the earliest known lunar calendar of 360 days which dates back to 4236 B.C. Furthermore in many of the early number systems to be discussed later numbers such as 60 and

360 play an important rôle strongly suggesting that the systems were first used as calendrical aide-memoires.

Of the earliest methods of recording we can of course know nothing for certain but one can imagine stones being placed in a pile, notches being made on trees, or marks on stone. As men did this the pattern of thirties would become perceptually clear, though of course these activities would not require the invention of a number language but rather some form of tally or matching operation. The visual representation was adequate for the purpose and there was no need to name each notch or scratch as it was made. Authorities are in agreement that the earliest number systems had their origin in calendrical records of this kind, and it seems likely that we have here the first step towards counting, which is fundamentally the same kind of matching operation, though with the use of words instead of notches on trees.

For the origin of number names we have to look at a rather different kind of experience. The hunter had few possessions and presumably there were few occasions when he would be concerned with the size of a collection. Most likely the first ideas of quantity were associated with such patterns as the family group, the number of young produced by animals in the neighbourhood, certain familiar star-clusters in the sky which were helpful in locating position. Sometimes when one went to collect the berries there were only a few, sometimes there were many, but there was no need to be exact about the number. Sometimes four pups were born and only two survived. There might be one daughter in the family and three sons. Names for very small collections occur early in the history of language but even after the shepherd had taken over from the hunter numbering seems to have played a minor rôle in the life of man. To quote Smith: 'For a long time after the advent of man such simple names as two or three were sufficient for all purposes not met by nouns of multitude like lot, heap, crowd, school, pack or flock.'[1]

---

[1] Smith, D. E., *History of Mathematics*. Vol. I.

The particular names he invented are of special interest since they give some indication of the thought processes behind his numbering. They were not as yet part of a number series and had a merely descriptive use. They were qualitative as well as quantitative in intention and the same term served to express the nature of the objects as well as their numerical character. For this reason there might be several words to denote a collection of two, each of which was only used in relation to one particular kind of collection, but no word to correspond with our two which can be applied to any collection of this size. In our language we retain a piece of this history in the number of words we have for designating a collection of two, viz. brace, couple, set, twin, team, pair; each of which can only be correctly used to refer to collections of certain kinds. There are many examples of this kind in the language of primitive societies and they occur before the language contains any words for collections which can be applied generally. For example, the natives of Fiji use different words to describe a group of canoes, a group of fish, and a collection of coconuts.

This suggests that an awareness of groups which can be distinguished by qualities including size occurs earlier than the recognition of exact quantities as such. Cassirer says of this mode of thinking, 'Individual groups are recognized and differentiated by some individual characteristic; in so far as one can speak of "number", it appears not in the form of a specific measured magnitude, but as *a kind of concrete numerical gestalt*, an intuitive quality adhering *to a totally unarticulated general impression of quantity*'.[1] (The italics are mine.)

For this reason early languages developed no universal number words applicable to all enumerable objects, such as those which form our number series. Because number was seen as a quality of things, of the same order as other qualities such as colour, texture, shape and the like, there had to be as many diverse numbers and groups of numbers as there were different classes of things.

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[1] Cassirer, E., *The Philosophy of Symbolic Forms*. Vol. I.

This characteristic of early number words persisted even after they had become independent of the designation of things and attributes, and prevented them from becoming universally applicable. To quote Cassirer again:

> Not every number applies to everything; it does not yet express abstract multiplicity as such, but expresses the mode, class and form of a concrete multiplicity. In the American Indian languages, for example, different groups of numerals are used to designate persons or things, animate or inanimate objects. Or a different group of numerals may be used to designate fishes or pelts, standing, lying or sitting objects. The Moanu Islanders have different sets of numbers from one to ten for coconuts or men, spirits and animals, trees, canoes and villages, houses, poles and plantations. In the Tsimshian language of British Columbia there are special series of numerals for counting flat objects and animals, round objects and time intervals, men, boats, long objects and measurements; in other, neighbouring languages, the differentiation of the various series of numerals goes even further and seems almost unlimited. As we see, enumeration is by no means orientated towards 'homogeneity'. *The tendency of language is rather to subordinate the quantitative difference to the generic difference expressed in its classifications and to modify the expression of quantitative difference accordingly.* This tendency is evident even where language has progressed to the point of using universal numerals, but where each numeral is still followed by a specific *determinative* indicating the particular class to which the group belongs. Seen intuitively and concretely, there is obviously a great difference between the gathering of men into a 'group' and the gathering of stones into a 'heap', between a 'row' of resting objects or a 'swarm' of moving objects, etc. Language seeks to retain such classifications and shadings in the choice of its collective terms and in the regularity with which it combines such words with actual numerals. In the Malayo-Polynesian languages, for example, numerals are not directly attached to substantives, but instead to certain determinatives which are required to classify the group. The term for 'five horses' is literally 'horses, five tails', for 'four stones' it is 'stones, four round bodies', etc. In the Mexican languages, the expression of number and of the enumerated objects is likewise followed by a term of group classification, which differs for example for round and cylindrical objects like eggs and beans, for long rows of persons, things, walls and furrows, etc. (Italics are mine.)

Children's thinking seems to have something in common with what is being described here. The little girl who learns to say two in connection with her shoes, Stephen (see page 6) who only knew five as a certain pattern of dots, and Catherine (see page 6) who knew she couldn't be three because she was four, are all using number names without full recognition of their quantitative significance. The twoness of two and the fiveness of five have not yet been abstracted and these numbers are not, therefore, free for general application to any collection of two or five respectively, nor are they recognized as occupying positions in a number series. These particular developments are of major importance in the growth of number understanding for which every teacher should be watching.

Many number words in early languages indicate a rather different origin, and perhaps represent a slightly more advanced level of thinking. In everyday experience there occur certain natural groupings which, because of their size-constancy, come to be used as model groups against which other groups can be matched. It is clear from the records we have of pictographic script that such natural groupings as a pair of wings, a three-leafed clover, four-legged animals and five fingers came to be used in this way. One simply matched one's collection against one of these familiar groupings and used the name of the latter to denote its size. In this way man could arrive at the notion of plurality and differences in sizes of groups without needing to count them. His groups were small, and matching could be done visually, perhaps with the aid of finger-pointing. When a number word has been created it serves the same function as the original model. The Gestalt psychologist, Wertheimer, made a special study of the numerical concepts of primitive peoples and in discussing the function of these natural groupings he points out that this method of evaluating groups by matching does not contribute to the development of abstract numbers.

To us it seems a great advantage to be able to abstract from the natural relationships of things. It is this facility of thought which we must thank for many technical achievements in our civilization. But there are others whose thinking must be described thus: Wherever there is no natural relationship, no vividly concrete and relevant connection amongst things themselves, there is also no logical connection, nor is any logical manipulation of these things possible. In contrast to this stands our kind of thinking whose logic tends in this direction: 'Everything can be counted', and 'All things can be combined in and-summation'.[1]

Dantzig, who also discusses this phase of development, says:

> A rudimentary number sense, not greater in scope than that possessed by birds, was the nucleus from which the number concept grew. And there is little doubt that, left to this direct number perception, man would have advanced no further in the art of reckoning than the birds did. But through a series of remarkable circumstances man has learned to aid his extremely limited perception of number by an artifice which was destined to exert a tremendous influence on his future life. This artifice is counting, and it is to counting that we owe the extraordinary progress which we have made in expressing our universe in terms of number. . . . It is counting that consolidated the concrete and therefore heterogeneous notion of plurality, so characteristic of primitive man, into the homogeneous abstract number concept, which made mathematics possible.[2]

This quotation prepares the way for the next stage in our story and for the recognition that it is an extremely important one. With the development of more highly-organized societies and the growth of settled communities came the need to measure and assess quantities more accurately and to deal more precisely with larger collections. Out of this need came counting. The static models which had been used in matching small collections no longer served, since it became necessary to split up a large group and deal with it in stages. Similarly the names for small collections described earlier were no longer adequate.

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[1] Wertheimer, M., 'Numbers and Numerical Concepts in Primitive Peoples.' *Source Book of Gestalt Psychology.*

[2] Dantzig, T., *Number: The Language of Science.*

At first, simple matching or tallying operations were carried out using the same methods used for recording the passing of time—putting out stones, making marks on wood, clay or sand, tying knots in string, etc., and probably fairly early in this era man discovered the handy tally he carried about with him in the shape of his own body. But whereas man can hold up four fingers to communicate the number of spears he is willing to exchange for his neighbour's bullock, how is he to communicate regarding the size of groups larger than twenty which he might denote by using both hands and both feet? He might call his neighbour over to see the result of his tally operation in the form of marks he has made in the sand but this is not a very convenient way of carrying on business and the development of a language of number words and symbols became a necessity.

A study of some of the early examples of the series of number words which were invented shows clearly that at first they were not used in the way we use the words in mathematical language. The first counting operations consisted merely in designating differences in sizes of collections by transferring them to something else, most often to parts of his body, but quite often to groups of things like stones. Thus, according to Smith:

> In Malay and Aztec the number names mean literally one stone, two stones, three stones, and so on; while the Niuès of the Southern Pacific use 'one fruit, two fruits, three fruits', and the Javans 'one grain, two grains, three grains'; all these being relics of the concrete stage of counting. When a Zulu wishes to express the number six he says 'taking a thumb', meaning he has counted all the fingers on the left hand and has begun with the thumb of the right. For seven he says 'he pointed' meaning that he has reached the finger used in pointing.[1]

Exactly the same thing happened in measuring continuous quantity. In many instances the first units used were parts of the body and the foot unit retained in our linear system is an example of this, though a standardized unit is now used. The ell has its roots in the Anglo-Saxon eln-boga meaning elbow, and the thumb unit came into our language through the Latin pollex to the French pouce to the English inch. In course of time these terms lose their primitive meaning and we come to use them as abstract measures. In just the same way the primitive words used in counting discrete quantities were tied at first to concrete groups, either to the items originally being counted as in the example from Fiji given earlier, or to the units being used in the tallying, be they stones, fruits, grains or fingers.

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[1] Smith, D. E., *History of Mathematics*. Vol. I.

We seem then still not to have reached the stage in the story when man is using numbers as abstractions in the way we do when we are counting and calculating, but a new element has become involved which proves itself as playing a very important part in this particular development. This part of the story can be very instructive for the teacher of young children and merits special attention. Because in counting operations of this kind man is using his hands and fingers in various ways as well as his eyes, his sensory experience during the activity is enormously enriched. Proprioceptive sensation (i.e. sensations yielded by the sense organs situated in body muscles and tendons) is added to visual sensation, which means that all the information yielded by movements is added to what is given by sight, and out of this experience the significance of order becomes established. Cassirer, in an extremely interesting discussion explores the importance of this development more fully than we shall do here and suggests that because of the bodily action involved in primitive counting operations, 'all numerical concepts are purely mimetic hand concepts or other body concepts before they become verbal concepts.'[1] He quotes in support of this theory many examples from primitive languages where 'the numerals do not so much designate objective attributes or relations of objects as embody certain directives for the bodily gesture of counting. They are terms and indices for positions of the hands or fingers, and are often couched in the imperative form of the verb. In Sotha, for example, the word "five" means literally "complete the hand", and that for six "jump", i.e. "jump to the other hand."'

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[1] Cassirer, E., *Philosophy of Symbolic Forms*. Vol. I.

SENIOR BAS... COLLEGE

As Cassirer points out we do not have in these early systems a general organization of numerical concepts:

> But one thing is accomplished: a very definite order is observed in passing from one member of a manifold to another, even though this manifold is determined in a purely sensuous way. In the act of counting, one part of the body does not follow another arbitarily, the right hand follows the left, the foot follows the hand, the neck, breast, shoulder, follow the hands and feet, in accordance with a scheme of succession which is conventional, to be sure, but is in any case strictly observed. The institution of such a scheme, though far from exhausting the content of what more highly developed thought understands by 'number', nevertheless provides its indispensable groundwork. . . . In apprehending material objects not only according to what they individually and immediately *are*, but according to the manner in which they are *ordered*, the spirit begins to advance from the concretion of objects to the concretion of acts; and through acts, the acts of combination and differentiation which it performs, it will ultimately arrive at the new 'intellectual' principle of number.

As Cassirer has indicated, the importance of this stage in the history of numbering cannot be over-estimated. Because primitive counting methods became increasingly orientated to the body, consciousness of order in progression could become increasingly articulate as man passed from one part of his body to another in an habitual order in accordance with the convention of his group. Perhaps it is not too much to say that man was feeling his way into a number system. Certainly bodily actions communicated to the brain through proprioceptive sensations were now conveying information of a very important order.

The rôle of proprioceptive experience in learning has not received sufficient attention in educational literature, indeed many people seem to assume that all experience is mediated through our five external sense organs. The messages received by the brain from the activity of the interoceptors situated in the walls of internal body organs and the proprioceptors situated in the muscles and tendons yield information about

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events which is at least as important as information received from the extroceptors. His five external sense organs can tell the child a great deal about the world around him but they cannot for instance tell him about lightness and heaviness, roughness and smoothness, roundness and flatness, rhythm and order. The only way in which he can become aware of these dimensions is through movement information relayed to the central nervous system in terms of proprioceptive sensations. Lightness and heaviness are experienced in terms of muscle effort needed to lift an object, roughness and smoothness in terms of the resistance offered to movement by various surfaces, rhythm and order in terms of a series of sensations.

The suggestion that children should be allowed to move about and play with materials of various kinds is often justified in terms of physical growth needs and emotional satisfactions. What should be understood is that they can learn things about themselves, and about the world around them, through moving about and handling things that they can learn in no other way, since at these times they are receiving all the vital information which motor sensation alone can yield. These ideas are discussed more fully in chapters 4 and 9.

The story of the development of number is obviously not yet complete, but before proceeding further it may be as well to consider the points so far made which are specially relevant for us as teachers. Firstly we have noticed how number awareness grows out of certain experiences in daily life in which some particular quality in the experience demands man's attention because it is relevant to his personal needs. In this way through his dependence upon seasonal changes, he became aware of sequences in natural events and devised ways of recording them. Similarly, in certain situations quantity is important and as soon as a simple matching operation is incapable of dealing with the quantities he needs to assess he learns to combine a matching with an ordering operation and to develop the skill of counting.

Surely children's learning also proceeds most satisfactorily

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when it is felt by them to be relevant to their own needs. Too often we are guilty of trying to teach them things before they are really ready because they are going to need them at some future date. This failure in timing our educational programme is one of the major causes of frustration for both children and teachers. However it is certainly true that very young children show an appreciation of sequences related to events which effect his well-being and his need of rhythm and order is so strong that he becomes extremely disturbed when it is upset. Every mother soon learns how important it is to her baby not to make sudden changes in the routine. It is also true that little children soon learn that certain things belong together; his shoes, his gloves, his spoon and fork. This is the beginning of awareness of groups. These are the basic experiences, the rudiments of awareness which make all later developments possible, so that in a very real sense it can be said that number education begins in the first few months of life. Miss Thyra Smith has explored the significance of these early experiences very fully in her book called *Number* which makes it unnecessary to do so here.

We have noticed, too, how far man can get in the art of reckoning without leaving the concrete stage in his thinking. Perhaps we do not always recognize how long after the child is using the vocabulary of number he is still at this stage in his thinking. We have been too ready to assume that when he uses a word he intends the same meaning as we should intend were we to use it. If, however, we observe carefully the situations in which he uses number words we shall discover this is by no means the case. Piaget's[1] experimental study of the development of the number concept shows clearly that the children he tested did not reach the abstract stage till far later than is usually expected. These findings are, of course, extremely important for the teacher and will be discussed in more detail later.

The final point to be stressed is the suggestion that the first instinctive bodily responses to rhythm, sequence and pattern in the immediate environment are the beginning of the story. These provide an experience of what Cassirer[1] called '*organic space and time*', an experience probably comparable to that which gives to such creatures as bees, ants and birds of passage the power of orientation which they display to such a remarkable degree. Man, however, is able by reason of his many highly-developed sense organs, and his power of organizing his perceptions, to arrive at a perception of spatial and temporal relations which Cassirer terms '*perceptual space and time*'. In time these perceptions are translated into symbols which enable man to think about these experiences when they are not actually present. The language of words and signs is part of this symbolism, and the language of number is an extremely important part of this. As soon as this language has become truly symbolic, that is to say as soon as it can be used in thought to take the place of perceptual images, then man can arrive at the idea of *abstract space and time*. This achievement was not reached by men in the primitive societies which we have been discussing, nor is it reached in the early years of childhood. It was, however, achievement of this degree of abstraction which made possible the developments described in the following chapter.

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[1] Piaget, J., *The Child's Conception of Number*.

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[1] Cassirer, E., *The Philosophy of Symbolic Forms*. Vol. I.

CHAPTER THREE

THE GROWTH OF NUMBER SYSTEMS

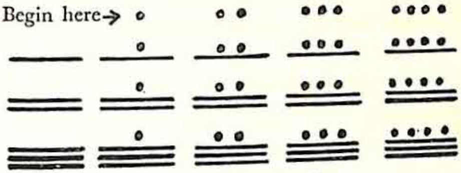
As noted in the previous chapter, out of primitive methods of reckoning a very important concept developed which is basic in any number system and in all mathematical thinking—the concept of order. In using the body for counting a conventional scheme must be observed within the social group to make communication possible. In some societies the fingers of the left hand were used first, then those of the right. In some systems the counter then proceeded to use the toes of his left and right feet, and in some, other parts of the body were also called into play. In others, man would use a sign to indicate a handful and then begin to use the fingers again. Thus there occurred quite early the use of a system of positions, but it is important to note that at this stage the ability to observe an order did not imply the use of a number scale, since at this time man was using his number words descriptively.

Another important element which influenced the systematization of number was the fact that as man began to deal with larger quantities the idea of grouping naturally developed. Some number came to be recognized as the radix or base of counting, ten being the most usual, probably due to the common habit of using the fingers for reckoning. Ten was not the first radix to be adopted, however, and Smith[1] suggests that 'very likely two was first used, as in the case of those natives of Queensland who count "one, two, two and one, two twos, much."' He also offers evidence for scales based on three, four and five. The origin of the scale of twelve is of particular

[1] Smith, D. E., *History of Mathematics*. Vol. I.

interest to us since it is preserved in our measuring system. According to Smith 'There is reason to believe that the scale of twelve, the duodecimal scale, was favoured in prehistoric times in various parts of the world, but chiefly in relation to measurements', and he suggests that it is its divisibility by two, three and four which made it attractive. The choice of base in some systems seems to have been influenced by the fact that they were first used in calendar-reckoning. The Maya and Sumerian systems illustrate this.

The Maya System of time-reckoning is one of the earliest known to us and provides an interesting example of the way in which a number system develops out of the use which numbers are put to in any civilization.



Values from 1–19.

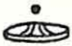
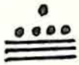
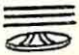
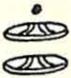
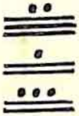
Mayan Numerals					
Values	20	39	300	360	6353
Interpretation in our notation	1.20 0	1.20 19	15.20 0	1.360 0 0	17.360 11.20 13

Fig. 1. MAYAN NUMERALS

Such evidence as we have suggests that the Maya did not use their number system for computation as did the Sumerians, Babylonians and Egyptians, but simply as a calendrical aide-memoire. This explains some curious features in their system. They counted on a base of twenty and were able to write any number using only three signs: a dot, a dash, and a kind of oval or lozenge-shape. The dot represented one, the dash five, and Fig. 1 shows one variant of the numerals up to nineteen, reading from left to right. The lozenge represented twenty and always occurs with other symbols having the effect of multiplying them by twenty as in the symbols for twenty and three hundred shown in the illustration. When a second lozenge was added, however, it multiplied by eighteen and not by twenty. Thus the lozenge acts as a kind of placing symbol rather like our zero sign, though the number base is uneven proceeding from 1 : 20 : 360 : 7200 and so on. The reason for this complication becomes clear when one remembers the use to which the system was put. The Maya calendar was a moon-calendar consisting of 360 days, making this number an important halting place in the system.

As can be seen from the numerals in the illustration the system makes use of place value, the values being placed above each other in order of magnitude instead of alongside each other as in our system. In the first example the dot is in the second position and has the value of twenty; in the second example nineteen is in the first position and the single dot in the second making a total of thirty-nine; while in the third example the fifteen being in the second position becomes three hundred. The other two examples show symbols in the third position which multiplies their values by three hundred and sixty. As already mentioned, the lozenge fulfils a similar function to our zero in that it serves to indicate magnitudes by positions, but it could not be used consistently throughout because of the variation in the base. Maybe if this feature had not been present the Maya would eventually have arrived at a system similar to our own. It needs little imagination to see how clumsy their system would have been had it been needed for purposes of computation. But, according to Smith:

There is no evidence in any extant record that it was used for purposes of computation, its use in the texts being merely to express the time elapsing between dates. The fact, however, that the pebble and the rod are apparently the basal elements in the writing of numbers leads us to feel that we have in these numerals clear evidence of the early use of an abacus.

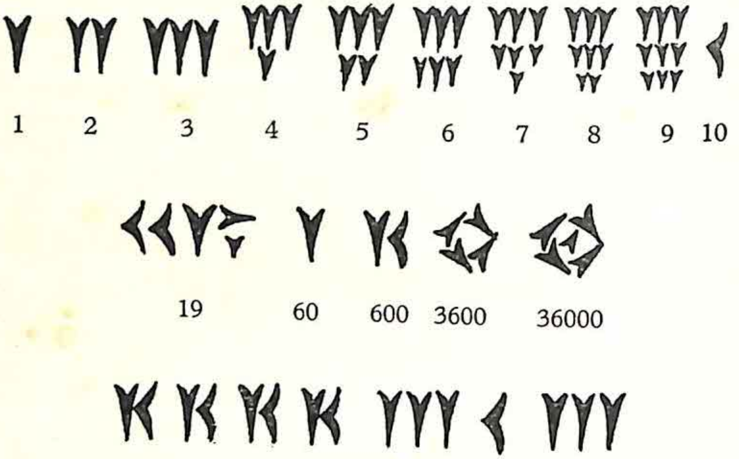
The oldest known systematic uses of written numerals in computation are those of the Sumerians and the Egyptians, both dating from about 3500 B.C. Each used a base of ten, though with some inconsistencies as will be seen. The numerals up to nine were collections of strokes, and special symbols were used for collections of ten, a hundred, etc.

The Sumerian system and the numerals used are known to us through the discovery by archaeologists of the clay tablets on which they worked, many of which can be seen in the British Museum. They show that by the twentieth century B.C. their merchants were familiar with bills, receipts, accounts and systems of measure, but there is also evidence that the Sumerians used some kind of yearly calendar as early as 4700 B.C.

Here again there is evidence to suggest that finger-reckoning and calendar-making influenced the development of a system which is partly decimal and partly sexagesimal. The Sumerians counted in units from one to ten and in tens to sixty, but starting from sixty, they built onto the decimal system a system based on the unit of sixty or upon its factors of six or twelve. The following diagram shows how the system built up compared with our own:—

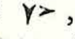
1	1
10	10
10×10	10×10
$(10 \times 10) \times 10$	$(10 \times 6) \times 10$
$(10 \times 10 \times 10) \times 10$	$(10 \times 6 \times 10) \times 6$

From archaeological evidence it is clear that the Babylonians in general adopted the business methods, the astronomy, the calendar, the measures, and the numerals of the Sumerians. Most of the evidence comes from some 50,000 tablets and cylinders unearthed at Nippur, and Fig. 2 shows the symbols used. They



The number 2593

Fig. 2. BABYLONIAN NUMERALS

were made with a wedge-shaped reed on damp clay and are not always easy to interpret. The numerals for the values from one to ten are shown; but, as the symbol for one also stood for ten and in general for 60^n, and the symbol for ten also stood for 10.60^n, the context of the symbol determined the value to be taken. It is of interest to notice that the Babylonians made a slight use of the subtractive principle as did the Romans much later, e.g. XIX is the equivalent for XX – I. The Babylonian symbol for minus was , thus the symbol for 19 shown in Fig. 2 could be read as 20 minus 1. They also used the ten as a multiplying symbol as in the numerals for 600 and 36000 shown in Fig. 2. As in the case of the Maya system one finds many variants on the basic system, particularly in the symbols denoting large numbers, since there was no compelling force to standardize them. A circle was often used for 10 and crescent-shapes for units, particularly in the early manuscripts. Moreover the circle was also used for zero, at least to the extent that they employed it to represent the absence of number, but, according to Smith, it played little part in their system of notation. Despite the complications of the system and the appearance of large numbers, such as that shown in Fig. 2, it is clear from the tablets and cylinders found at Nippur that by about 2400 B.C. the Babylonians had developed considerable skill in computation and mensuration in spite of the awkward numeral system they used. Calculations include multiplication and division tables, tables of squares and square roots, geometric progressions and the use of unit fractions.

Turning now to the Egyptians; Smith says, 'Whatever claims may properly be made for the antiquity of mathematics in various other countries, claims of even greater validity can justly be made for the science in Egypt.' The third millennium B.C. saw the rapid development of practical engineering, culminating in the astonishing achievement of the Great Pyramid, which was also a considerable mathematical achievement.

In the second millennium B.C., during the reign of Amenhotep III, an extensive system of irrigation was carried out which must have necessitated a considerable knowledge of levelling, surveying and mensuration. Probably, too, in this reign was written the original of the oldest mathematical treatise now extant, the Rhind papyrus, so named after its discoverer. Parts of this can be seen in the British Museum. It includes a table on the resolution of fractions, calculations involving the multiplication and division of fractions, a treatise on volume and cubic content, areas and division of land, a method for calculating angle of slope, and many other mathematical problems showing that, as Yeldham says, 'by the Twelfth Dynasty, which is the date of the earlier work from which the scribe is copying, the mathematician was already able to work out any problem which he was likely to meet in ordinary life'.[1]

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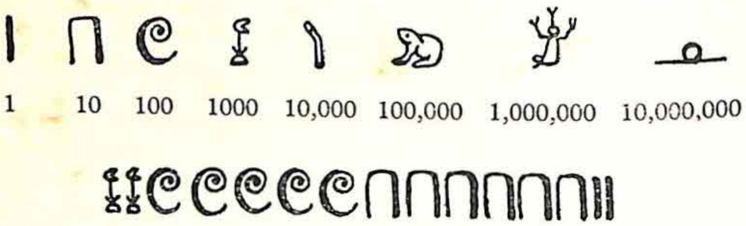
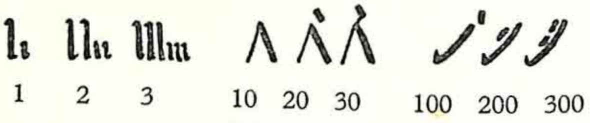
[1] Yeldham, F. A. *The Story of Reckoning in the Middle Ages.*

These achievements seem the more remarkable when their numeral systems are examined, since they seem so extremely cumbersome compared with our system. The Egyptians had four materials on which they could record events: stone from quarries worked along parts of the Nile valley; papyrus, a kind of paper made from strips of pulp of the water reed; wood and pieces of pottery. When writing on stone they used hieroglyphics, commonly written from right to left, but sometimes in the reverse direction. Fig. 3 shows examples of these, and also of the hieratic numerals used on papyrus, wood and pottery. Another form, the demotic numerals, was used in the Rhind papyrus. The only method of writing large numbers was the method of repetition of symbols of different magnitudes as shown in Fig. 3 and one can only wonder at the skill of the mathematicians who advanced so far with such a system.



The number 2,562
Hieroglyphic.



Hieratic.

Fig. 3. EGYPTIAN NUMERALS

The hieroglyphic in Fig. 4 is taken from Breasted who gives the following interpretation:

> Above is the falcon. The falcon is the sign for the king and it leads the head of a man by a cord. Behind the head are six lotus leaves (each the sign for 1000) growing out of the ground to which the head is attached. Below is a single-barbed harpoon and a little rectangle which is the sign for a lake. The whole tells the picture story that 'the Falcon king led captive 6000 men from the land of the Harpoon lake'.[1]

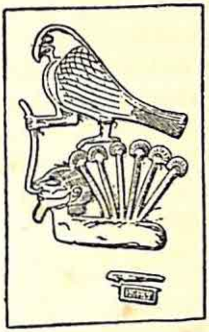


Fig. 4. AN EGYPTIAN HIEROGLYPHIC

Coming nearer to our own times the Greeks and Romans developed systems which were basically of the same kind as those of the Babylonians and Egyptians, using a base of ten with special symbols to indicate units, tens, hundreds, thousands, etc. These are shown in Figs. 5 and 6 with the methods used in writing large numbers which show clearly the difficulty in systems of this kind when compared with the same number written in Arabic numerals.

What is still lacking is the system of positions which controls the decimal system. The Roman notation, for example, employs the system of repetition, using separate symbols for each decimal magnitude: units I, tens X, hundreds C, thousands M or CIↃ in the earlier forms. When any number contains more than one of any of these magnitudes the symbol must be repeated as many times as necessary and though the additional symbols, L for 50 and D for 500, curtail the length somewhat, e.g. MMDCXXIII, one would be somewhat daunted by an Arithmetic book which contained sums written in numerals of this order.

In our system we need only to employ our nine integers and a zero in writing any number, however large, since the position of any integer in the series of numerals is an indication of its magnitude. 8 may stand for eight units, eight tens, eight hundreds, etc., according to where it occurs in the series. Thus the number given above in Roman numerals becomes 2623.

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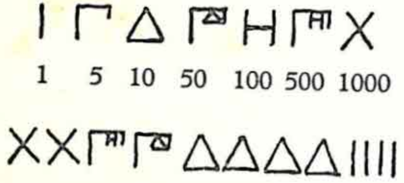
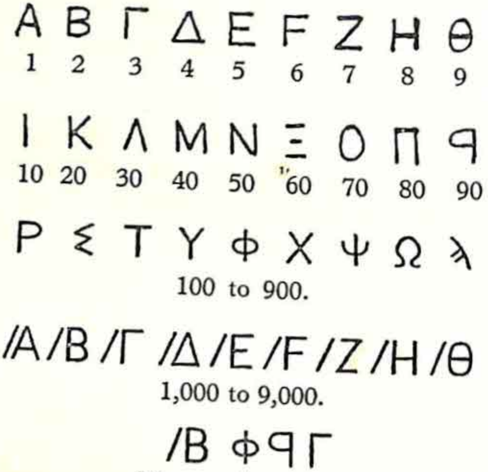
[1] Breasted, *A Brief History of Ancient Times*.



The number 2594.

Ionic.

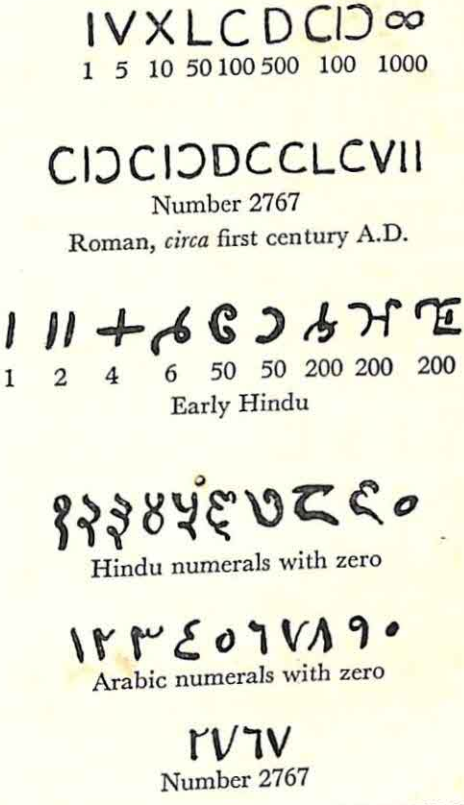


The number 2593.

Attic.

Fig. 5. GREEK NUMERALS

The possibility of using the principle of position without involving the principle of repetition depends on the use of a special symbol for absence of number. There must be some kind of symbol in each position to hold the position of the others. Suppose, for instance, that the Roman number given above



Number 2767

Roman, *circa* first century A.D.

Early Hindu

Hindu numerals with zero

Arabic numerals with zero

Number 2767

Fig. 6. ROMAN, HINDU AND ARABIC NUMERALS

had been MMDIII, we should not write it as 253 but as 2503, since we have a symbol to indicate that there are no tens. Both the Maya and the Sumerians had a zero symbol to indicate an absence of number but they never learned to use it in this way.

The story of how man came to use a symbol to indicate an empty column, and the subsequent elaboration of a number system with place-value, is one of the most interesting parts of our story, and certainly one of the most important, since it had far-reaching consequences for the development of Arithmetic and its use in scientific measurement. For the first time it became possible to express all numbers by means of ten symbols, each symbol receiving a value of position as well as an absolute value. The idea is so simple and familiar to us now that we ignore its merit, though it may come as a surprise to some readers that the system was not adopted in this country until the seventeenth century A.D., in spite of the fact that it was known for several centuries before this. The inventor of the symbol for zero is unknown, but by the sixth century A.D., it was part of the Hindu-Arabic system of numerals and it was almost certainly suggested by the abacus which was used for reckoning at this time.

Very early in the history of counting man devised methods for dealing with numbers larger than he could hold in his hand. Probably at first he set out pebbles or sticks in grooves of sand, successively representing units, tens, hundreds, etc., or whatever were the multiples in the system he was using. In course of time counting frames were used for this purpose, and abacus is the name by which these are known. According to Smith:

> In earliest time the word 'abacus' seems to have referred to a table covered with sand or fine dust, the figures being drawn with a stylus and the marks being erased with the fingers when necessary . . . the dust abacus finally gave place to a ruled table upon which small disks or counters were arranged in lines to indicate numbers. This form (shown in Fig. 7) was in common use in Europe until the opening of the seventeenth century and persisted in various localities until a much later date.[1]

Other forms of abaci have grooves or rods for movable balls or disks as in the examples shown in Fig. 8. The rod form is still in use in Russia, China, Japan and parts of Arabia, though it is gradually being replaced by the modern calculating machine. In this form the beads on each rod carry different decimal orders of magnitude: units, tens, hundreds, etc., according to the position they occupy. The design of the frame varies considerably. For example, in the Chinese abacus shown in Fig. 8, each bead in the upper part of the frame carries the value of five and is moved down to the centre in a calculation, while those in the lower part carry the value of one and are moved up

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[1] Smith, D. E., *The History of Mathematics*, Vol. II.

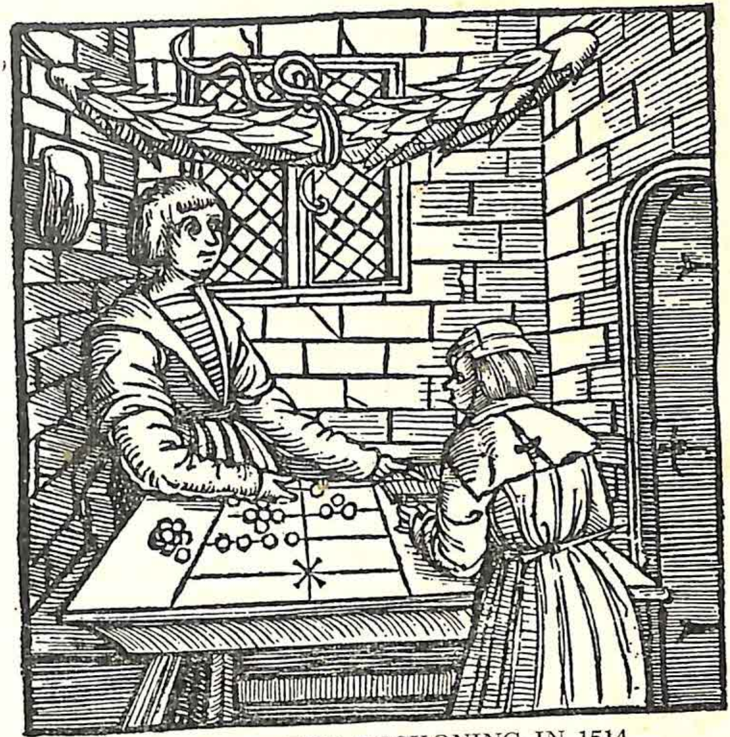
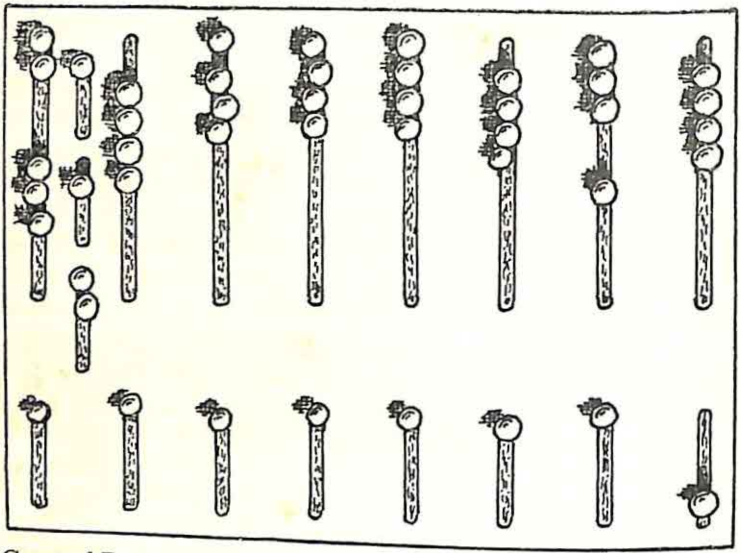
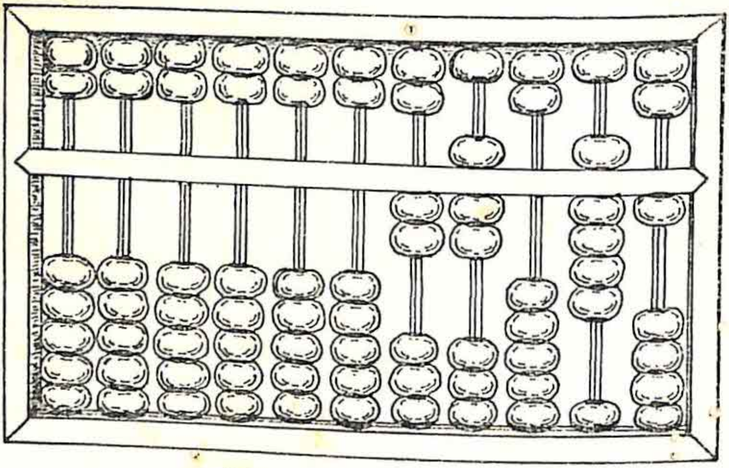


Fig. 7. COUNTER RECKONING IN 1514



Grooved Roman abacus in bronze. Date uncertain. *British Museum*.



Modern Chinese abacus.

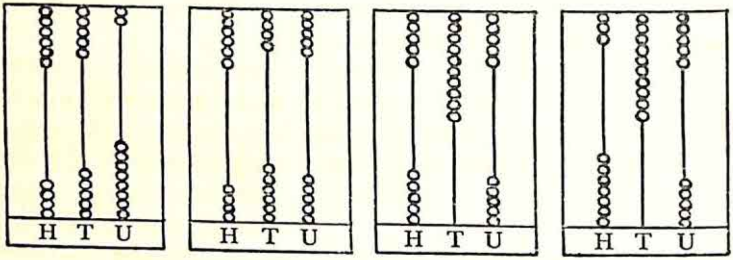
Fig. 8. ROMAN AND CHINESE ABACI

to the centre. The number indicated on the frame in Fig. 8 is therefore 27091 in our notation.

In this country various firms now supply abacal frames for use in schools, with three or four rods each carrying nine or ten beads. Before beginning a calculation all the beads are pushed to the top of the frame then, if the task is to add 247 to 458, eight beads will be brought down in the units column, five in the tens column, and four in the hundreds column. The remaining two beads in the units column are now brought down and these ten pushed back while one bead is brought down in the tens column. Five beads are now brought down in the units column and the operator proceeds to deal with the tens column, bringing down four beads. This gives a total of ten beads which are all pushed back while one bead is brought down in the hundreds column. This leaves the tens position empty, and all that remains is to bring down two more beads in the hundreds column. The results of these operations are shown in Fig. 9 which also serves to illustrate the function of the zero symbol.

In all the number systems previously discussed the orders of magnitude have been indicated by using different symbols, e.g. the Roman I, X, C, M, etc. The Hindus, however, thought of the ingenious device of having a symbol to indicate any column that was empty. By using this, the positional value of any other numeral is obvious, since one can see at a glance to which column it belongs. Thus in the sum we have been doing instead of writing the answer as CCCCCCCV, as would have been done in some Roman systems, we can write 705, the 0 having the effect of telling us that the 7 refers to seven hundreds and the 5 to five units. In other words we have no need of special symbols to indicate orders of magnitude, nine symbols and a zero being adequate for all purposes.Fig. 10 shows clearly the advantages of the system.

The Hindus called the symbol for zero the śūnyabindi (generally shortened to śūnya). The word literally means 'empty' or 'void'. This passes over into Arabic as sifr or as-sifr and the English word cipher and French chiffre are derived



458 458+7 458+47 458+247=705

Fig. 9. ADDING 247 TO 458 ON THE ABACUS

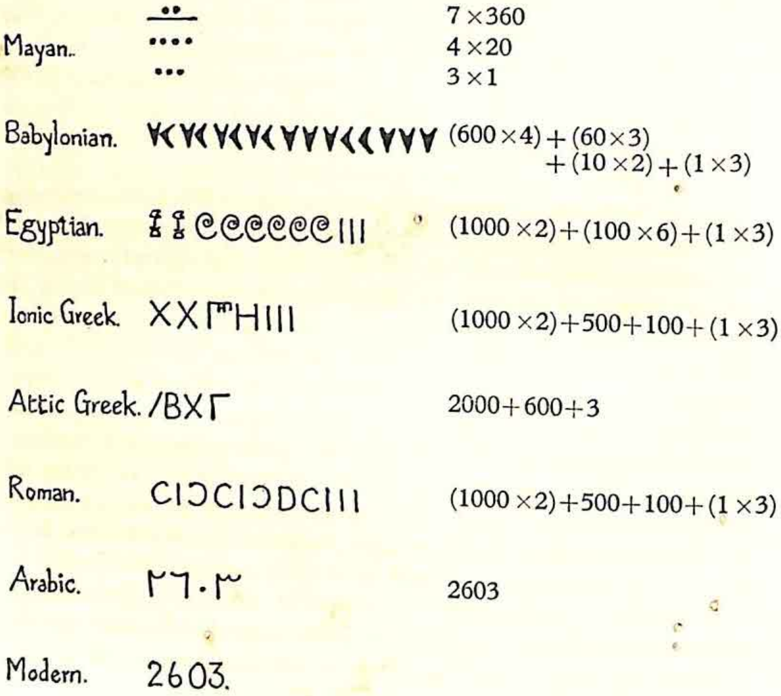


Fig. 10. NUMBER 2603 IN VARIOUS SYSTEMS

from this. From as-sifr has also come zephirum and finally zero found in Calandri's Arithmetic in 1491.

It is not known exactly when the Hindu-Arabic numerals were introduced into Europe but they probably came in through the traders. What is certain is that early in the twelfth century books appeared in Europe explaining the art of Hindu reckoning, and what is surprising is that their obvious advantage over other forms did not lead to their immediate adoption. Leonardo Fibonacci of Pisa, one of the outstanding mathematicians of this period, wrote a treatise on the numerals explaining how they could be used in computations, but the spread of such knowledge was bound to be slow since communication was largely by word of mouth. There must, however, have been other factors of resistance at work to explain the centuries that went by before the system was generally adopted.

Fig. 11, taken from one of the earliest encyclopaedias to be compiled, depicts the cleavage that had developed among mathematicians by the end of the fifteenth century. The abacists preferred to stick to the methods of their ancestors; the algorists favoured the use of the algorithms of the Hindu-Arabic system. The prejudice of the abacists may be explained partly by their loyalty to the Christian faith and consequent resistance to any heathen invention, but there are other reasons which probably help to explain why the numerals did not attract more immediate attention. Cheap paper was not available and pencils date only from the sixteenth century. Furthermore, modern methods of multiplying and dividing were not yet invented and counters were handy and quite adequate for calculating. However, in course of time the Hindu-Arabic numerals gradually replaced the Roman forms throughout Europe until by the seventeenth century they were in common use. It is interesting, however, to note that they are still unknown in some parts of the world where abacal reckoning is common practice though due to Western influence, particularly on the commercial life in the Far East, their adoption is becoming more world-wide.

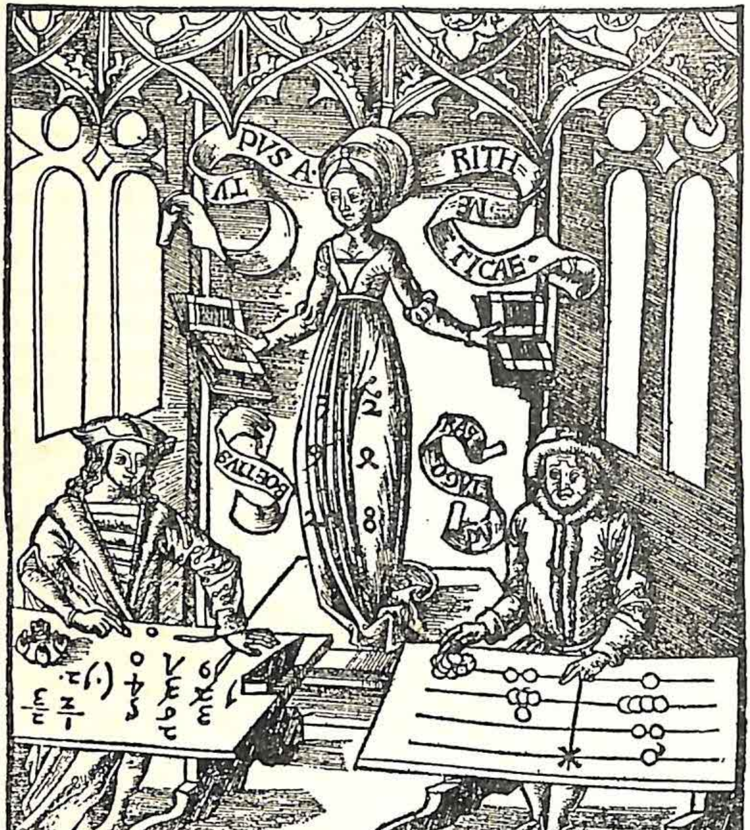


Fig. 11. ARITHMETICAE WITH ABACIST AND ALGORIST

The story of the growth of modern mathematics can be found elsewhere and need not be told in detail here. Galileo set the pattern for modern scientific thought by defining the goals of scientific inquiry and the rôle of mathematics in achieving them. In so doing he paved the way for the great metaphysical systems elaborated by Descartes, Newton, Spinoza and others. Galileo believed the universe to be mathematical in structure and behaviour, acting in accordance with inexorable and immutable laws. It was in this pattern of thinking that modern science was born and the course of its development has been profoundly influenced by a philosophy which affirms the mathematical design of nature.

This development not only made possible the truly remarkable achievements of modern science but it brought its own dangers, one of these being the tendency to reduce every variable to an item in a scale of measurement. Pascal was one of the earliest to recognize this danger and Cassirer comments on his contribution in these words:

> There are things which because of their subtlety and their infinite variety defy every attempt at logical analysis. And if there is anything in the world that we have to treat in this second way, it is the mind of man. What characterizes man is the richness and subtlety, the variety and versatility of his nature. Hence mathematics can never become the instrument of a true doctrine of man, of a philosophy of anthropology. It is ridiculous to speak of man as if he were a geometrical proposition. . . . All the so-called definitions of man are nothing but airy speculation so long as they are not based upon and confirmed by our experience of man. There is no other way to know man than to understand his life and conduct. But what we find here defies every attempt at inclusion within a single and simple formula. Contradiction is the very element of human existence. Man has no 'nature'—no simple or homogeneous being. He is a strange mixture of being and nonbeing. His place is between these two opposite poles.[1]

As a tool of knowledge mathematics is now indispensable but one must beware of becoming the slave of a mere tool. The science of psychology cannot be absolved from guilt in this respect, and psychometrics has not always been used with discrimination in investigating variables in human behaviour. Moreover the use made of statistics by the social sciences

[1] Cassirer, E., *An Essay on Man.*

generally tends to over-simplify and cover up just those variations from the norm which make up the pattern of society as we know it. A. N. Whitehead was fully aware of the weakness in the materialistic scientific philosophy which modern man has inherited from the seventeenth and eighteenth centuries:

'. . . the whole concept of materialism only applies to very abstract entities, the products of logical discernment. The concrete enduring entities are organisms, so that the plan of the *whole* influences the very characters of the various subordinate organisms which enter into it. In the case of an animal, the mental states enter into the plan of the total organism and thus modify the successive subordinate organisms until the ultimate smallest organisms, such as electrons, are reached. Thus an electron within a living body is different from an electron outside it, by reason of the plan of the body. The electron blindly runs either within or without the body; but it runs within the body in accordance with its character within the body; that is to say, in accordance with the general plan of the body, and this plan includes its mental state'. This principle is general throughout nature and for Whitehead its recognition involved 'the abandonment of traditional scientific materialism and the substitution of an alternative doctrine of organism'.[1]

It is essential to maintain this sense of organism in all psychological and educational inquiry. Whatever aspects of human behaviour we may choose to observe we must always be aware that we are observing something which is taking place in a context—biological and social. There is always an interaction between inner and outer activity. This is what makes the study of any aspect of human learning such a very complicated matter. We have to understand as well as we may the nature of the inner mental events which correspond to outward observable behaviour and the nature of the interaction between the individual and his environment.

In this discussion we may seem to have gone a long way from our main theme but it is more relevant than may appear at first sight. In the mid-twentieth century we live in a scientific world and much of the new knowledge which is being gained can only be expressed in the language of mathematics. This means that every individual who wishes to play a responsible rôle in our society must learn at least some of the language, and it is vitally important that the nation's children receive a better mathematical education than has been obtained hitherto. Changes are already being made and these changes include a new approach to the basic education in the Infant School. These changes, however, are not necessary only because of the need for an increasing number of scientists. A good many ideas put across to the public in advertisement and propaganda are conveyed in the language of statistics and the intelligent citizen should know how to evaluate the facts communicated in this form.

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[1] Whitehead, A. N., *Science and the Modern World.*

The story of number recorded in these two chapters covers a long period in the history of mankind. This is something the educator should remember. Today the little child hears the language of number and picks up many of the words as early as any others. This does not mean that he endows them with a mathematical meaning, and he must be allowed time to feel his way into the language as did his forebears. Man felt his way into it through his own bodily manipulation of his environment. Children do likewise and must be given the opportunity as well as time to do this. Naturally children do not need as long as early man since the accumulated wisdom of the race can be transmitted to them through education, but they can only absorb that part for which they are ready by reason of mental development and experience. In these days we are less prone to forget this than heretofore but we are still inclined to rely too heavily on the value of verbal instruction. This is true of much of the teaching given in our Primary Schools today, and is certainly true of Arithmetic teaching where the unfortunate results can be detected more readily.

The story of the invention of the zero sign and its function in a number system with place-value suggests one example where correct timing of instruction might prevent some of the difficulties which children and even some adults have with this symbol. Caroline obviously did not understand its purpose and maybe difficulties of this kind would not arise if the symbol were not introduced until its function could be explained with the aid of some kind of abacus. It is not really needed until children are ready to write large numbers and do simple calculations with them. In the majority of cases this will not be till they reach the Junior School. Of course much younger children count up to numbers beyond 100 and are able to write them down, but this does not mean that they understand or need to understand why some numbers include a nought and others do not. Small children may also encounter its use in scoring games and may learn to write the symbol when they have failed to score a point. When, however, they are ready to do calculations with large numbers they must learn to use the 0 as a spacing symbol, something which maintains the value of other symbols in a number. This can be understood most easily with the aid of an abacus, which one would like to see in use in all Junior Schools.

CHAPTER FOUR

THE BEGINNINGS OF UNDERSTANDING

As has been shown in earlier chapters, Number is a language which enables man to classify certain of the variables he observes in the physical events he experiences and to define relationships of a particular kind. If this is so then clearly it is necessary to inquire how the child becomes aware of these aspects of experience and arrives at a need to name and define them.

Number words are some of the many things which the little child picks up by imitation from the adults among whom he is living, yet, as was seen in chapter 1, it is clear from observation of the way he uses them that they do not carry for him the meaning the adults intend when they use them. It seems obvious that they cannot carry such meanings until the child has noticed these particular features in his experience, and has digested something at least of their significance; until in fact they have come to have some meaning for him.

Actually the roots of this understanding begin to grow very early in childhood, just as soon as the little baby is able to respond selectively to his environment. The problem for the infant, as for any living organism, is that of maintaining a state of equilibrium within himself, and between himself and the external world; a state of affairs which is absolutely essential for his survival. However, he is not endowed to anything like the same extent as other animal organisms with instincts to control his behaviour. He can, from a very early age, exercise his power of choice and uses this to perform accommodating movements in response to the sensory impressions he is receiving. At first the impressions are confused and result in diffuse interpretations and responses but, as the nervous system develops, responses become increasingly selective and purposeful.

Piaget[1] finds in these early sensori—motor—affective responses the beginnings of the operation of intelligence and it is pertinent to this inquiry to ask, 'What kind of picture of himself and the world is built up by the child at this early stage in his development? How is this picture related to the responses he is capable of making? How is it related to the pictures he will be able to build at a later stage in his development, and what kind of influence has it on language development?'

The new-born baby is forced into action as soon as he arrives—he must breathe and very soon he must feed and excrete. All these activities arouse excitement in sensory nerve endings situated inside the body—interoceptors in the visceral organs, and proprioceptors situated in the muscles and tendons. These are the first communications to the brain—sensations of movement and touch which contribute primordial knowledge—body-consciousness without which there can be no life. Traditionally, as we have already noticed, it is usual to emphasize the importance of the exteroceptors which are stimulated by energies external to the body and yield information about the world outside. In the history of the individual they are of secondary importance in the first few weeks of life. The organism must become conscious of itself before it can become conscious of itself in relation to what is outside itself, though it is likely that these two acts of consciousness take place almost simultaneously, and once body-consciousness and world-consciousness are established each will feed the other.

There are good reasons for believing that at first all three sources of sensory information are present in the brain in an indifferentiated, or partially differentiated form, and that perception of environmental objects is inextricably fused with the motion, position and feeling state of the individual. Even at a later stage, when sense modalities are distinguishable from each other they still work together to provide information about events. Visual and auditory information must be pooled with proprioceptive information as to the position and motion of the receptor surfaces if effective contact with the outside world is to be maintained.

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[1] Piaget, J., *The Origin of Intelligence in the Child.*

Many investigators agree as to the vital rôle played by proprioceptive experience in these early stages in development. For example, Sherrington suggested many years ago that mind, recognizable mind, seems to have arisen in connection with the motor act:

> It would seem to be the motor-act under urge-to-live which has been the cradle of the mind. The motor act, mechanically integrating the individual, would seem to have started mind on its road to recognizability. . . . As motor integration proceeds, mind proceeds with it, the servant of an 'urge' seeking satisfaction.[1]

Changes both within and outside the body are something we are living with every second we breathe, and living consists in numerous adjustments to these changes, ranging from gross motor acts, such as turning the head away from a bright light, to the finest repatterning movements of the nervous system. But every movement, however slight, causes modifications in the inner environment. For every movement made by the body a corresponding movement occurs in the central nervous system, which is thus being patterned and repatterned by every action. In this way, through a life of active response, the organism is building up within itself a model of the environment with which it is interacting; an inner map of events which enables it in course of time to anticipate external events and to interpret phenomena by reference to its inner model. The physiological processes involved in this activity tend in time to group themselves into organizations each with a distinctive function. These mental organizations are usually known as schemata, the name given to them by Head,[1] and of these the body-schema is of primary importance. The body-schema is present due to the fact that the cortex of the brain registers the relative positions of the various parts of the body at any moment. This registration, or schema, acts as a reference for the body when any change is intended. It gives an indication of how much effort will be needed, and which part of the body must be moved first to affect the desired change. In an article in *The Lancet*, Purdon Martin[2] indicates the important rôle played by the body schema:

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[1] Sherrington, C., *Man on His Nature*. 2nd edition.

> Out of proprioceptive and visual impressions we build up a body-image. We cannot maintain consciousness without this. . . . On this image, on the patterning of certain cerebral activity which we develop in our early years, we become so dependent that without it we cannot survive as sentient beings.

Throughout life these inner models are being modified and changed or restructured by every event which affects the organism in any way. They are the store houses of knowledge, acting as reference centres for all behavioural responses and, at the same time, remaining open to modification, strengthening and enrichment by these responses. Thus cognizing, whether at the perceptual or conceptual level, is a structuring activity in which the whole organism is involved, and this is true from the beginning of life.

This basic structuring process has begun long before language is possible, which means that language, when it does occur, develops within a framework which patterns language itself in the same way as it patterns everything else. Movements provide the first vocabulary of man—it is his displacements, accommodations, assimilations, and adjustments to the world which have told him what the world is like, and since these have been built into the brain-patterning they provide the raw material with which language must work.

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[1] Head, H., *Aphasia and Kindred Disorders.*

[2] Purdon Martin, *Consciousness and its disturbances.*

This account should make clear how tremendously vital to all later development are the child's early modes of response. Every time he moves, and he is moving all the time, he is adding something to his knowledge of himself and the world outside himself. He comes to terms with his environment by learning the operations necessary to produce desirable changes and in so doing he discovers what his body feels like, what it can do, its power to modify events. He is, so to speak, building into himself what Professor Meredith calls 'operational universals' which are reflected in his habitual modes of response and in the skills he develops. Moreover, he discovers something about the properties of the things on which he operates. Long before he has the vocabulary to describe them he knows something about hardness and softness, stillness and movement, heaviness and lightness, smooth and rough, in terms of the degree of resistance he meets in his relationship with things. These notions are incorporated into his operational universals. His world is taking shape. Later he will be able to use words to describe variations in the qualities of experience such as those mentioned above, and later still he will be able to define degrees of variation in mathematical terms. Here, however, are the basic experiences upon which all later, more advanced understanding rests. Here then is the beginning of number education, because experience is forcing him into awareness of relationships.

Earlier in this book the statement was made that space and time are the two fundamental realities with which man has to come to terms, and in this early pre-lingual behaviour it is possible to distinguish signs that this adjustment is being made by the very young child. Notions of space are probably the earliest to emerge and certainly they are basic in the problems of adjustment already described. The body-schema is after all only a map in which all the various parts of the body are spatially related. As Werner[1] puts it, 'Space is originally one aspect of the child's consciousness of his own body. Space literally ends at the periphery of the physical being, it is enclosed, so to speak, within the skin of the child.' However, this state of affairs changes rapidly with the knowledge he accumulates as a result of his exploratory movements in relation to things in the environment. The mouth is the first organ of exploration and a good deal of spatial information is acquired as an object is sucked. Presently the child is able to see things which are not touching him—the rattle, his mother's hair, etc., and through his reaching out, touching and grasping, objects become known and relationships of touching and not touching begin to have meaning. With an increase in the power of visual discrimination, and in the possibilities of movement, space gradually expands, and the relationship between his own body and things located outside begin to be understood. He shows appreciation of the movements necessary to reach a desired object. Near and far, high and low, under and above, now become meaningful in terms of the kind and degree of effort needed to reach one's goal. These notions, like those mentioned earlier, will later be defined in the spatial language of topology and geometry, but here basic experience is laying the necessary foundations for fuller understanding and conceptualizing at a later period in development.

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[1] Werner, H., *Comparative Psychology of Mental Development.*

One has only to watch for a short time the explorations and play activities of a little child emerging from babyhood to begin to wonder at the amount of understanding he has acquired. He uses different movements to pick up a biscuit, to pick up a ball, to push his train and to get himself upstairs. He soon knows when part of his train is missing and where to put away his toys. Here is the beginning of spatial knowledge which will be discussed more fully in chapter 9.

As Piaget[1] has pointed out:

> In a sense it can be said of time, as of space, that it is already given in every elementary perception; every perception lasts, just as every perception extends. But the first duration is as removed from time properly socalled as is the extension of sensation in organized space.

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[1] Piaget, J., *The Child's Construction of Reality.*

Time, like space, is constructed little by little and involves the elaboration of a system of relations.

The first intuitions of time emerge, as do those of space, in the context of the little child's motor-affective response to his environment. At a very early age the baby shows himself capable of co-ordinating his movements in time and performing certain acts before others in regular order. For example, he opens his mouth and seeks contact with the nipple before beginning to suck. Such acts are, of course, performed and stabilized into habits without conscious perception of a temporal order; nevertheless, the sequence is patterned in cerebral activity. Similarly, the child experiences a series of perceptions without perceiving sequence as such. In this way, time as a sequence of events is being patterned into the mental life of the child because it is inherent in the life he is living. Moreover, since affectivity in the form of effort, desire, expectation, feelings of dissatisfaction followed by feelings of satisfaction, are part of the pattern of these early experiences, awareness of duration and of sequences of states is further reinforced.

The little child who has reached the stage when he can begin to apply familiar means to new situations is forced to attend to the order of events in the external world as well as to the ordering of his own actions. Time is now experienced not only in terms of events occurring one after the other within oneself, but is also experienced as an objective series. No parent would need to be convinced about the very young child's appreciation of rhythmic sequences and perhaps some would have stories to tell similar to the following: A friend of mine was watching the game of 'This little pig went to market' being played with her cousin's baby of about ten months of age. Each toe was squeezed at the appropriate moment, but on one occasion the mother's attention was distracted, and she failed to squeeze the last toe. The baby, evidently recognizing that the familiar sequence had not been completed, leant slowly forward and squeezed it himself. Then he looked up and laughed with delight. It will be some years before this basic experience of sequence and order becomes definable in mathematical terms, but the basic notions of slow and fast, before and after, now and soon, are understood fairly early in life in terms of events as they happen to the little child. Many one-and-a-half to two year olds find meaning in such phrases as 'Wait for Mummy', 'Daddy is coming soon', 'Be quick'.

In due time, of course, the child struggles towards the generalization from experience underlying adult-usage of temporal terms. Stern[1] has a rather nice sequence of records illustrating this phase of development in their daughter, Hilde. At 3; 5 speaking of a coming journey home, Hilde said, 'If we go home, then it will be today.' At 4;3 she expressed it this way, 'We will pack today and start yesterday.' But at 5;1 she dictated to her mother the following message for her father who was away from home: 'Today we are cooking nice things', and added an explanation for her mother, 'Today—do you know because it is today father will know at once it was yesterday' (i.e. 'When he receives the letter on the next day).' Stern comments, 'What confusion reigns for a long time in the child's little head before he grasps that the ever-moving present turns today into yesterday and tomorrow into today.' Nevertheless, it seems to me that three-and-a-half-year-old Hilde knew from experience that there is a difference between the here and now of today and the there and then of tomorrow.

What is being suggested here is that the first awareness of relations which will later be defined and categorized in the language of number are embedded in the early dawnings of spatial and temporal awareness. Experience of space includes experience of magnitude and multitude; experience of time is experience of sequence and order. Counting is a means whereby a multitude can be assessed with precision, whereby a magnitude can be measured, and whereby a sequence can be annotated. But experience of these manifolds is possible long before there is any felt need to operate upon them in this way, and these experiences provide the basis for an understanding of the operations when they become necessary. Indeed, it is the increasing awareness of the significance of these variables, and the use to which they can be put, which brings the child to the stage of naming them, and translating perceptual awareness into a conceptualized understanding.

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[1] Stern, W., *The Psychology of Early Childhood.*

The kind of knowledge described in this chapter might be defined as 'operational knowledge'. It takes the child quite a long way in dealing with life as he experiences it, but the ability to verbalize this knowledge and the subsequent act of conceptualizing will take him much further. The function of the educator is twofold: to put the child in the way of extending his operational knowledge, and to help him to acquire a language in which he can think and talk about it.

Note that here, as in so many other matters, it is the parent who is the first educator. Number education really begins in the first year of life and those of us who meet children for the first time when they come to school have but to share in the process. Those who ignore this fact do a grave disservice to children. We should be prepared to take trouble to try to discover what our children already know, and how they look at their world and themselves in relation to it. Only so can we hope to preserve the essential continuity of the educational process which began at birth. Our task is to build on to what is already known, to enrich meanings, to enlarge horizons. We do not start from scratch, only parents have this privilege and responsibility.

CHAPTER FIVE

THE ORIGINS OF SYMBOLISM

THE language of words and other forms of symbolism which the child hears, sees and imitates are the carriers of man's thought and feeling about his experience. They enable him to move forward to new levels of understanding, adding to his symbolism as he goes, and the achievements of modern physicists bear testimony to the power of language, in this case the language of mathematics.

But every individual has to enter for himself into an understanding of the symbolism, if he is to share in the cultural life of his group. This is one of the ends of education and is operating in all its many different forms, as much in the means the priests use to initiate the young into the myth and rituals which for them enshrine religious truths, as in the classroom lessons in chemistry, local government, literature, or mathematics.

One of the difficulties inherent in communication is that one cannot always be sure that the next man is using symbols in the same way as we are. When one considers words for instance it is obvious that they are endowed with meanings by their users. The truth is that, like Humpty Dumpty, when we use a word it has just that meaning we intend when we use it—no more and no less. Moreover when a word is addressed to us we interpret by attaching to it the meaning we should intend were we using it. In the absence of any evidence to the contrary this seems the sensible thing to do, nevertheless it is a source of failure in communication which may sometimes have serious consequences. Resort to a dictionary is not necessarily helpful since all it gives is the meaning most commonly attached to the word in the particular cultural group for which the dictionary is intended, and there is no guarantee that the speaker has this meaning in mind at the time of speaking. In the context of adult communication experience has taught us the necessity of inquiring into meanings and it is sometimes possible to establish this at a verbal level. This is not always possible with children and more often one looks for a clue in the whole context in which the word is used, including any action which may accompany it. When a five-year-old says 'a hundred' while he is counting his beads we may infer that he means 'a lot' by the words and actions which occur before and after. Similarly it was possible to infer what Stephen (see page 6) meant when he used the word 'five', and it was clearly more limited than the meaning intended by his teacher.

One of the characteristic features of a young child's response to quantity which Piaget records is that he makes global judgments regarding the size of groups, the amount of liquid in a glass vessel, or the amount of material in a lump of clay. For example, he may correctly name a group as four when the items occur close to each other but if they are then spread out he will say there are more because they occupy more space. Similarly if the child watches all the liquid from one glass vessel poured into another of a different shape he will think there is now more liquid if the level is higher because the vessel is narrower, and vice versa, if the vessel is broader than the original one. The same kind of response is also obtained if the child watches a ball of clay being rolled out into a long snake or flattened into a pancake. Here, too, basing his judgment on spatial considerations, he will say there is now more clay. Clearly he has not yet grasped the idea of conservation of quantity, and the number words he uses do not stand for invariant properties of sets. For the adult a group of four is always four, no matter how other properties and relations within the group change. Piaget's studies show clearly that numbers have a fluidity in children's thinking which bears no relation to the stability and fixity implied by the concept of invariance, and anyone who cares to watch little children's use of number words will find confirmation of Piaget's findings.

Numbers, however, could not be used in computation in the way mathematicians are accustomed to use them unless they carry this property of invariance, and it is interesting to speculate on the meaning which $2 + 3 = 5$ might hold for a child at Stephen's level of thinking. It is clear that we cannot allow the kind of licence which Humpty Dumpty claimed for himself when it comes to number language since it enshrines for the mathematician certain basic concepts on which the whole mathematical system rests. There is a special language of words, symbols and signs to be learned along with the basic concepts for which they stand. Our business is to ensure that the child secures this language for his efficient use in dealing with situations which call for it.

This will take time and the process cannot be hurried without endangering later development. Somehow or other the verbal patterns he picks up and enjoys using, and the written symbols he also picks up and enjoys making patterns with, have got to become linked with the understanding he is gaining through the kind of experience considered in the previous chapter. When this begins to happen the two will develop together to enrich his understanding.

This is not to minimize the very important rôle which language plays in mental development from its beginning. It makes possible the transformation of direct experience of people and things into symbolism. Something is given a name and in this process certain qualities of the thing in question have been abstracted, classified and stored for identification purposes on subsequent occasions.

In order to understand how this coming together of language and experience occurs, it will be necessary to inquire into the nature of the rôle of language in mental development and in particular to try and understand the connection between language and thought in the life of intellectual man. This is something which should be appreciated by *all* teachers, but certainly those who have to do with young children when language ability is developing should try to appreciate the extraordinary importance of what is happening, since they have a significant part to play in this development.

It is difficult, perhaps impossible for those of us who acquired language in the normal way to appreciate the extent of the handicap to all mental activity, where there is an absence of language. Perhaps it will help if we try to enter imaginatively into what the experience of the rediscovery of language meant to Helen Keller, who at the age of three had lost the power of sight and hearing.

This is how Helen's teacher describes the event in letters to a friend:

April 5, 1887.

I must write you a line this morning because something very important has happened. Helen has taken the second great step in her education. She has learned that *everything has a name, and that the manual alphabet is the key to everything she wants to know.*

In a previous letter I think I wrote you that 'mug' and 'milk' had given Helen more trouble than all the rest. She confused the nouns with the verb 'drink'. She didn't know the word for 'drink', but went through the pantomime of drinking whenever she spelled 'mug' or 'milk'. This morning, while she was washing, she wanted to know the name for 'water'. When she wants to know the name of anything, she points to it and pats my hand. I spelled 'w-a-t-e-r' and thought no more about it until after breakfast. Then it occurred to me that with the help of this new word I might succeed in straightening out the 'mug-milk' difficulty. We went out to the pump-house, and I made Helen hold her mug under the spout while I pumped. As the cold water gushed forth, filling the mug, I spelled 'w-a-t-e-r' in Helen's free hand. The word coming so close upon the sensation of cold water rushing over her hand seemed to startle her. She dropped the mug and stood as one transfixed. A new light came into her face. She spelled 'water' several times. Then she dropped on the ground and asked for its name and pointed to the pump and the trellis, and suddenly turning round she asked for my name. I spelled 'Teacher'. Just then the nurse brought Helen's little sister into the pump-house,

> and Helen spelled 'baby' and pointed to the nurse. All the way back to the house she was highly excited, and learned the name of every object she touched, so that in a few hours she had added thirty new words to her vocabulary. Here are some of them: Door, open, shut, give, go, come, and a great many more. P.S. I didn't finish my letter in time to get it posted last night; so I shall add a line. Helen got up this morning like a radiant fairy. She has flitted from object to object, asking the name of everything and kissing me for very gladness. Last night when I got in bed, she stole into my arms of her own accord and kissed me for the first time, and I thought my heart would burst, so full was it of joy.
>
> April 10, 1887.
>
> I see an improvement in Helen from day to day, almost from hour to hour. Everything must have a name now. Wherever we go, she asks eagerly for the names of things she has not learned at home. She is anxious for her friends to spell, and eager to teach the letters to every one she meets. She drops the signs and pantomime she used before, as soon as she has words to supply their place, and the acquirement of a new word affords her the liveliest pleasure. And we notice that her face grows more expressive each day.[1]

And this is how Helen herself writes of what was for this little blind and deaf child an epoch-making discovery:

> It was a bright, clear spark from Teacher's soul that beat back the sooty flames of thwarted desire and temper in little Helen's no-world. That spark was the word 'water'. . . . Her body was growing, but her mind was chained in darkness as the spirit of fire within the flint. But, at last, on April 5th, 1880, almost exactly a month after her arrival in Tuscombia, Annie (Helen's teacher) reached Phantom's (Helen's name for herself at this stage) consciousness with the word 'water'. This happened at the well-house. Phantom had a mug in her hand and while she held it under the spout Annie pumped water into it, and as it gushed over the hand that held the mug she kept spelling w-a-t-e-r into the other hand. Suddenly Phantom understood the meaning of the word, and her mind began to flutter tiny wings of flame. Caught up in the first joy she had felt since her illness, she

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[1] Keller, H., *The Story of My Life.*

reached out for new words to identify whatever objects she touched. Spark after spark of meaning flew through her mind until her heart was warmed and affection was born. From the well-house there walked two enraptured beings calling each other 'Helen' and 'Teacher'. Surely such moments of delight contain a fuller life than an eternity of darkness.

It was not until after the episode at the well-house that Phantom felt an impulse towards something definite—learning the names of objects she desired or touched. Even then it was only a rudimentary impulse.

What happened at the well-house was that the nothingness vanished, but Phantom was not yet in a real world. She associated words correctly with objects she touched, such as 'pump', 'ground', 'baby', 'Teacher', and she gave herself up to the joy of release from inability to express her physical wants. She was drawn to Teacher, not by any sense of obligation, but by the natural impulse of receiving from her finger-motion what her word-hunger craved, just as the infant reaches out to his mother's breast for his milk. She only thought the words she had learned and remembered them when she needed to use them. She did not reflect or try to describe anything to herself. But the first words which she understood were like the first effects of the warm beams that start the melting of winter snow, flake by flake, a patch here and another there. After she had learned many nouns, there came the adjectives, and the melting was more rapid. Finally Teacher dropped in the verbs, one by one, sometimes in groups, but for Helen there was no connection between the words, no imagination or shape or composition. Only gradually did she begin to ask questions of the simplest kind. She had not conceived such things as 'what', 'where', 'how', and 'why', and other word-pegs and hooks on which we hang our phrases, but as she acquired them and framed halting questions, the answers from Teacher's hand banished her isolation. Teacher talked to Helen all the time after that, and oh, how things changed from fragmentary groupings beneath that communicable hand-to-hand magic! It must have required Teacher's utmost ingenuity to guess what incoherent bits of thought were striving in the child's mind for expression. It must have wearied her eyes to read Helen's fast-flying fingers as the child became more and more inquisitive, trying to gain in months a vocabulary that she should have started five years before. Yet, with the scanty, elementary information

Helen could understand, every object she touched was transformed. Earth, air, and water were quickened by Teacher's creative hand, and Phantom disappeared as life tumbled upon Helen full of meaning—mother and father, baby Mildred, Cousin Leila and her small girls, the Negro children who had for years borne her wild ways and tried to create play for her, and were loving to her always. That flood of delight in restored companionship was the real wonder of those early days and not Helen's miscalled 'phenomenal' progress in capturing language as a fully formed instrument.[1]

As Helen herself says, the most important thing that happened to her as a result of her rediscovery of language was that it brought her into communion with those around her in a quite new way, but it also had the curious power of transforming the physical objects she learned to name. By virtue of being named they were somehow quickened into life and became full of meaning. It is this latter characteristic of language which we must try to understand.

It may be helpful at this point to analyse the distinction which can be made between the sounds uttered by animals and human babies, and sounds which constitute true language. The former are expressive of immediate feelings and desires and cannot be taken out of their sign-reference. True language only begins when sounds keep their reference when they are taken out of their original context—when they can be used to refer to something not physically present, and so used either in speech with others, or in reflective thinking with oneself. In other words, the sounds have become symbols which in our minds can stand for the actual experience to which they refer. Many authorities[2] agree in emphasizing that the primary function of language is to enable man to deal with experience symbolically in this way, and that the function of language as a means of communication with others is a later development.

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[1] Keller, H., *Teacher*.

[2] e.g. Ogden, C. K. and Richards, I. A., *The Meaning of Meaning*. Sapir, E. Whorf, B. L., see *Language in Culture*, edit. Hoijer. Lewis, M. M., *Infant Speech*.

Once this power to transform experience into symbols has been discovered there is a rapid increase of symbols and a tendency for all subsequent experience to be related to it. Susanne Langer,[1] has described the process in this way:

> The brain is not merely a great transmitter, a super switch-board; it is better likened to a great transformer. The current of experience that passes through it undergoes a change of character, not through the agency of the sense by which the perception entered, but by virtue of the primary use which is made of it immediately; it is sucked into the stream of symbols which constitutes a human mind. . . . The fact that the human brain is constantly carrying on a process of symbolic transformation of the experimental data that come to it causes it to be a veritable fountain of more or less spontaneous ideas. As all registered experience tends to terminate in action, it is only natural that a typically human function should require a typically human form of overt activity; and that is just what we find in *the sheer expression of ideas*. This is the activity of which beasts appear to have no need. And it accounts for just those traits in man which he does not hold in common with other animals—ritual, art, laughter, weeping, speech, superstition and scientific genius. . . . Speech is, in fact, the readiest active termination of that basic process in the human brain which may be called *symbolic transformation of experiences*. The fact that it makes elaborate communication with others possible comes at a somewhat later stage.

This is a very suggestive interpretation of the motivation of the speech of young children. Many research workers have commented on the essentially egocentric character of little children's speech. They pay little attention to the responses of others when they are chattering and will talk just as happily when no one is present as when they have an attentive listener. They appear just as contented to chat to a companion who does not understand them as to one who gives a desired response. As Susanne Langer suggests speech is at first simply part of the child's natural habit of acting out what is going on inside himself, part of the pattern of expressive activity. This kind of acting out is enormously important in helping him to come to terms with the world outside and himself in relation to it. It is one of the ways in which he comes to an understanding of what is happening to himself because of the experiences he is having. Through it he will discover pattern and order, or perhaps it might be more accurate to say is finding ways of patterning and ordering his experience.

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[1] Langer, S., *Philosophy in a New Key*.

The adults with whom the little child is living have a very important part to play at this point in the child's development. Though it is true that he is an egocentric little creature he nevertheless thrives on attention and there is plenty of evidence to support the statement that the child who is neglected or ignored will be retarded in language development compared with other children of the same age.

As all mothers know, children invent words of their own at this stage. Alan began to refer to water as 'blop-blop' and anything he wanted to eat as 'um-an-ah', each of which 'words' have interesting movement qualities, lending support to earlier remarks about the importance to the child of this aspect of his experience. Nevertheless it is by imitation that he acquires the habit of using sounds to name things and to refer to them when they are not present. This means that he needs models to imitate. He must hear sounds used in this way and meet with a responsive audience when he makes his first attempts to do the same thing.

Little children of the same age differ enormously among themselves with regard to the rate at which language develops and also with regard to the range and style of vocabulary; differences which seem to be related as much to the quality of social experience as to the development of innate mental capacities.

Susanne Langer's statement brings out a further very important point about language which is that the nature of the symbolism man develops determines to some extent the kind of ideas he can generate. Language itself influences our modes of learning. It determines to a very large extent what aspects of our experience we reflect upon and what aspects we ignore, since most of our thinking is undertaken with words. Sapir puts it this way:

> Human beings do not live in the objective world alone, nor alone in the world of social activity as ordinarily understood, but are very much at the mercy of the particular language which has become the medium of expression for their society. It is quite an illusion to imagine that one adjusts to reality essentially without the use of language, and that language is merely an incidental means of solving specific problems of communication or reflection. The fact of the matter is that the 'real world' is to a large extent unconsciously built up on the language habits of the group. . . . If one draws some dozen lines, for instance, of different shapes, one perceives them as divisible into such categories as 'straight', 'crooked', 'curved', and 'zig-zag', because of the classificatory suggestiveness of the linguistic terms themselves. We see and hear and otherwise experience very largely as we do because the language habits of our community predispose certain choices of interpretation.[1]

We had an opportunity to notice this phenomenon at work in an earlier chapter when we were examining the number concepts of primitive peoples. Different societies ordered, classified and categorized their experiences differently, developing their own number languages. These, once formed, tended to perpetuate characteristic ways of ordering and categorizing within each social group which persisted over many generations. These varied number languages illustrate the truth of Whorf's statement:

> We cut up and organize the flow of events as we do largely because, through our mother tongue, we are parties to an agreement to do so, not because nature itself is segmented in exactly that way for all to see. Languages differ not only in how they build their sentences but in how they break down nature to secure the elements to put in these sentences; or again, the linguistic system (in other words, the gram-

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[1] Sapir, E., quoted in *Language in Culture*, edit. Hoijer.

> mar) of each language is not merely a reproducing instrument for voicing ideas but rather is itself the shaper of ideas, the program and guide for the individual's mental activity, for his analysis of impressions, for his synthesis of his mental stock-in-trade. . . . We dissect nature along lines laid down by our native languages. The categories and types that we isolate from the world of phenomena we do not find there because they stare every observer in the face; on the contrary, the world is presented in a kaleidoscopic flux of impressions which has to be organized by our minds—and this means largely by the linguistic systems in our minds.[1]

Educationally, of course, this is an enormously important idea. If as teachers we are concerned that children learn to pay attention to some particular aspect of their experience, that they begin to think and reason about it, then it is essential that we give them the language through which they can do this. Moreover, since the particular vocabulary and language structure we give them will become the symbolism through which they come to comprehend and categorize the experience, we must give thought to the form and structure of the language we are handing on from the point of view of its categorizing possibilities.

This will be equally important whatever field of knowledge we are wanting children to enter, but it is perhaps easiest to illustrate with respect to the language of mathematics. The symbolism of mathematics teaches us to order and classify experiences in particular ways, e.g. according to size, shape, position, etc. It helps us to become aware of and to define relationships of a particular kind between events, it encourages us to group and order them in certain ways, and once we have learned the language we become predisposed to categorize experience within this framework, and to use the structures of the system as though they existed a priori.

This kind of development will only occur, however, if the language acquired is a true language. If the words and signs are acquired in the context of real concrete experiences, they can become symbols for ordering these experiences at the mental level as well as the concrete, standing in thought for the actual experiences, as we saw earlier. In this way the relations perceived in concrete experiences will become attached to the language, as well as the nature of the language drawing attention to particular relationships. If, however, the signs are taught in an arbitrary fashion they do not become useful symbols in this way, but remain as signs whose significance is wrapped in mystery. This is what happened all too often in so-called mathematical education in the past, as many teachers will testify.

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[1] Whorf, B. L., quoted in *Language in Culture*, edit. Hoijer.

Unless the language of number is communicated to the child in such a way that it forces him to think relationally about events in his experience then it will not serve as a firm basis for later, more advanced mathematical thinking. To give one example of what is meant here: the number series has a certain relational structure and any single number fact only has sense and meaning by virtue of its place in the whole system. Thorndike pointed out long ago that before a child can understand what he is doing in arithmetic he must understand the series meaning of a number, its collection meaning, its ratio meaning and its relational meaning. None of these terms make sense if one tries to think of the number in isolation from the series in which it occupies a distinctive position in relation to all the others.

If teachers are aware themselves of the relational character of the language of number then it is likely to come through in their teaching. They think this way themselves and will communicate this mode of thinking to children almost unconsciously. Very many teachers, however, were not themselves taught to think this way in their own education, hence they need to think out consciously and deliberately ways of promoting this kind of relational thinking in the children they teach. It is for these latter folk that this book is primarily intended.

It will be helpful to consider at this point some of the situations in which little children in our culture pattern hear and see the symbolism of number in use, situations which are close enough to their own needs and interests to provoke their attention and possibly imitation. Jimmie is eighteen months old, his little legs are becoming stronger every day and he can now get himself around the sitting-room by a curious mixture of body movements. His weight is increasing and now instead of carrying him upstairs in her arms, his mother holds him under the arm-pits and allows him to participate in the process of getting from one stair to the next. As he does this she chants 'One, two, three . . .'. How long will it be before Jimmie joins in the rhyme? Gillian is nearly two years old. She is learning to help her mother put away her playthings. As they put the bricks into their box mother says 'One, two, three . . .' One day Gillian will join in. It is not difficult to think of other situations concerned with family life in which the adult will use the number words for counting in the presence of the children and observation shows these words are imitated along with other aspects of the situation, such as movements, before there is any clear recognition of why they are appropriate. At first they are simply sounds you make when you are going upstairs or when you are putting away your bricks, or when you are doing up the buttons of your coat, or when you are putting out biscuits on the plate for tea. Which of such events is the first context for this particular piece of imitation will of course depend on variables in parents' behaviour and children's responsiveness, but it will be some time before he sees the connection between the various situations listed above and displays his counting skill in all of them. For this to happen an act of generalization must occur which may not be within the capacity of the child for some time after he has learned the words as a kind of rhyme you say in this or that situation.

Nevertheless this verbal accompaniment to action probably helps to reinforce the experience of order, of one thing coming after another, and thereby provides a basis for a later understanding of counting. Many children arrive at school at 4 or 5 years old having made the act of generalization referred to above, i.e. they recognize situations in which saying the number rhyme is appropriate. If one observes what they actually do on these occasions, however, it is clear that they do not yet understand the act of counting. For example, Mary is 'counting' the beads she has threaded on to a string—she may or may not begin with the first one on the string, she may or may not match each word in the number series with a bead in turn, but even if she does this with the first two or three beads in the row, it is unlikely that she will proceed far before the movements with which she touches the beads and the movements she makes to sound the words do not synchronize and it has ceased to be a one-to-one matching operation. This will be of no concern to her and need not be to us. Given time, and the experience of watching others counting and sometimes joining in, an understanding of the matching operation will develop in due course.

The following stories are illustrations of the kind of responses which suggest that this degree of understanding has been achieved. Four-year-old Paul often played with the beads, sorting them into groups according to colour. One day he came to tell his teacher 'There are four of every colour on this string.' Robert ($4\frac{1}{2}$) painted a picture into which he had put some steps with a man standing at the top. While his teacher was looking at it he said, 'I know how many steps there are, you know, because I can count them, watch me,' and then proceeded to count the steps one at a time to ten.

Little children also have experience of things occurring in groups. John has two hands, two gloves, two feet, two shoes, two arms, two sleeves in his coat. His Mummy has two eyes, and two strong arms. Research suggests that most two-year-olds are able to apprehend a group of two, and probably the history of his understanding of the use of the word two in this context is similar to the story told above, i.e. he first hears the word in connection with one of the pairs listed above, e.g. his shoes, and it becomes a word you say when presented with them. This does not mean of course that he is referring to the property of twoness when he uses the word in connection with them. He may in fact be thinking of their shininess, their redness, their straps or buttons. Again one has to wait for an act of generalization to take place, something which can only occur when the child has experienced a number of situations in which the property of two applies, and has reached the level of mental ability at which he can abstract the quality they all have in common.

The following stories illustrate the kind of responses which can occur when this act of generalization has been made. A little three-year-old in a Nursery School was cleaning teaspoons for the cook. After polishing two she pushed them across the table remarking, 'That's two.' Three-year-old Philip was taken into a concert hall. Pointing to the organ he said, 'Look, Daddy, that organ's got three keyboards. Mr. Osborne's has only two.' Stephen, aged three, noticing some medicine bottles which his teacher had put into the Wendy House for hospital play said, 'You've brought a lot of bottles, haven't you, Miss B?' 'Yes,' joined in Susan ($4\frac{3}{4}$), 'four brown ones and three white ones.'

Sometime between the age of four and five many children begin to show another kind of imitation. Letters and number symbols now appear in the children's drawings and paintings. Usually they seem at first to be nothing more than patterns put in anywhere on the page, often reversed and sometimes upside down. Later they are used more selectively. There may be a number on a train, a bus or a house. What meaning does the child give to these symbols? Are they anything more than names, chosen in an arbitrary manner? When one considers the situations in which little children find things numbered one realizes how difficult it will be for him to discover that numbers are often used to indicate position in a series. John goes to school on bus No. 5, but has no experience of buses numbered 1, 2, 3, 4 and 6. His house is No. 2 but No. 1 and No. 3 are on the other side of the road and perhaps never noticed by him. Goods in the shop windows have numbers on them but the difference between 2 for 6d., 2d. each, 2/– a dozen, and 2/11½ must be left to Mother to sort out. One day the connection between these numbers and the coins Mother gives the shopkeeper will be understood, but not until John is a little older, though if from time to time he is given pennies with which to buy his own sweeties he may reach some understanding of the situation earlier than children who do not have this opportunity for learning.

There are still other situations in which John will hear and see numbers in use. One day he is told that tomorrow he will be five years old, that this means he is becoming a big boy and will soon be going to school. Next day he receives presents to celebrate the occasion and at tea-time there is a special cake for him with five lighted candles on the top for him to blow out at an appropriate moment. One John I know found the information that he was now five rather difficult to digest and asked, 'If I'm five where are the other four?' His difficulty is the same as that of Catherine mentioned in Chapter I.

Five o'clock marks the beginning of Children's Hour on television. No doubt many children watch the hands moving on the clock face and know what five o'clock looks like, but what connection has this five with being five years old or having five fingers on one's hand?

A little while ago the writer was playing games with a group of five-year-olds in order to try and discover the meanings they attached to the number symbols. There were six houses cut out in cardboard to put in a row and the numerals one to six on little cards to be placed on the appropriate houses. Stephen knew the names given to these numerals and when asked to put No. 4 on the right house he counted from right to left up to four and placed the card on the fourth house. Then he took up the rest of the cards and placed them on the other houses in what appeared to be a random order. When asked 'Have they all got the right numbers?' he replied 'All got a number each—going to have a race 'cos they've got numbers like jockeys.'

These examples are perhaps enough to show that there is a difference between the little child's use of both verbal and written number symbols when he first picks them up by imitation, and their use by the older child as standing for number concepts with which he can undertake arithmetical calculations.

Nevertheless this is an important stage in the learning process. Very soon he will be able to recite the number words in their conventional order and to name the symbols correctly. This gives him a tool with which he can think about his experience—a symbolic language for naming, classifying and ordering things. Given time and opportunity he will endow the words and symbols with the meanings which adults give them, but it is important to recognize what a big step has to be taken before he can understand why the use of a number like five is appropriate in so many apparently disparate situations. Piaget's study of the growth of number concepts offers convincing proof that this achievement is not found in children until much later than has been generally assumed, and that its appearance must wait upon the maturation of mental abilities as well as on experience.

The moral for the teacher of young children seems to be that her job is to provide many kinds of experience which provoke them to order, to compare, to share, to group and to count. Moreover she must share in these experiences with them giving them the opportunity to enrich their number language by imitating her and watching the way she uses it.

Many of the playthings found in Nursery and Infant School classrooms offer opportunities for this kind of learning. Building bricks of different sizes and shapes may be ordered in different ways, and utensils for tea-parties in the Wendy House or for laying up tables for dinner provide experience of grouping and maybe for counting. Checking the number of paintbrushes, pairs of scissors, pencils, etc., to see that they are all collected and put away at the end of the play period provides a purposeful reason for counting. Clay, sand, and water play, particularly if there are utensils of various kinds to use, provide many situations which provoke comparison. Story periods when children listen to the story of the Three Bears, Snow White and the Seven Dwarfs, the Ten Little Nigger Boys, and the Seven Dancing Princesses, help to enrich number language and provoke interest in numbers.

It is not difficult to make children 'number conscious' and it is not unusual to find a group of children who spontaneously recognize and enjoy the quantitative aspects of experience. Several stories have already being given which illustrate this kind of interest and attention which may be found in a group of four- or five-year-olds, but the experience of the writer suggests that this is only found where some adult is playing a rôle as a stimulator of interest and a provider of challenging experiences.

For example these two incidents were reported by the same teacher with a group of four to five-year-olds. Julie drew a picture and then drawing the teacher over to see it, she said, 'Look, there are two suns, one this side and one that, so that makes two.' As she said this she pointed to each in turn and then held up two fingers. Robert drew a picture of a ship with funnels and portholes and said 'Look, it is a big ship because it has three funnels, one, two, three.' These pictures were later put on the wall with appropriate phrases underneath each, and no doubt this added to the children's interest.

Five-year-old Adrian entered another classroom one day with three friends. He took them over to a wall on which were hanging some number cards showing collections of various objects with the number symbols underneath. He stood his friends up against the cards saying, 'One, two, three.' Then he stood himself against card numbered four and said, 'Four soldiers,' Roland and a little girl in his class were making tarts for baking. Roland had made seven and the little girl five, when Jeffrey, who had been an interested spectator throughout, said, 'He has made two more than her.' On another occasion their teacher had put out seven jars of paint and six brushes on the table. Michael very soon came up to say that they needed one more brush. These children were used to helping their teacher with such things as checking the register and giving out milk and remarks like Stephen's were fairly frequent. 'There are thirty-two boys and girls at school today,' he told his teacher. When asked how he knew he replied, 'Well, there's thirty-one straws in the milk and Susan doesn't have any so that makes thirty-two.'

There are many games which five- and six-year-olds enjoy playing which encourage this kind of thinking as the following incidents suggest. John and Diane were playing with a fishing game. At the end John said to Diane, 'I've won because I've got thirteen fishes and you've only got nine.' Christine knocked a skittle down and said to Philip who was playing with her, 'Oh, I've knocked one down. That means there's five you can knock down.' Victor threw a five on his dice but did not move the counter on the home game. When asked why he explained 'Five is too many, I only need three to get home.' Terry and Stephen were playing with another home game when the teacher approached and Terry said, 'If I get all sixes and Stephen gets all ones I win.' Margaret also discovered something about the value of six through playing this game because Joan went up to the teacher to complain, 'Margaret's cheating —she's throwing it so that she gets a six every time.' Brian discovered something about the composition of ten as shown by his remark while playing skittles. 'There are seven skittles knocked down. I know that because there are three left standing up.' Howard was heard to say to the party in the Wendy House, 'I'll have to fill this kettle again, because it only makes three cups of tea and we need five.' Beryl, on another occasion was less accommodating! When four children arrived to play in the house she said, 'You can't all stay to tea. There's only enough for three cups in the teapot.'

It is hoped that the discussion in this and the previous chapter will suggest to the teacher of young children some ways in which she may provide for the beginnings of an understanding of number. There has in the past been a tendency to hurry children too quickly on to activities in which numbers are used in a more abstract way, and to confine number experience too narrowly to certain kinds of number apparatus, and it is necessary to examine very critically the provisions we are making for the development of an understanding of number relationships.

The little boy who could not recognize a group of five unless there was 'one in the middle' was living in a classroom where all the number apparatus was very carefully designed in the domino pattern series. There was a series of patterns on the wall with the symbols attached, the children played with dice and dominoes, patterned in the same way, and were given little tasks to do such as this one $\begin{smallmatrix}0&0\\0&0\end{smallmatrix} + {}^{0}0_{0} =$. This was making it difficult for him to arrive at a generalized concept of 'fiveness'. Moreover, in his experience, five always applied to a group, never to a position in a series. It would have been helpful to him had there also been up on the wall a line of houses to number, a string of beads to count, a ladder to move up and down; and in the toy cupboard a set of Tillich bricks for floor play.

The teacher must also examine the value of number tops, tunnel games and lotto. These are sometimes introduced with the idea of making number concrete but of course they do nothing of the kind. When little children are ready to operate with abstract numbers to do calculations expressed in equation form it is doubtful if they need the stimulus of a top to interest them in doing sums. Until they are ready, using a number top will contribute nothing to readiness if it is simply used to get two numbers down on paper to add together. Number tops can, however, be used instead of a dice for scoring games, when of course the situation is very different. Tunnel ball is sometimes used in the same way; if, however, the score is translated into something concrete like peas or straws and the game is played to see who wins the most, the experience is at a level at which the little child can operate.

The point being made here is that at the beginning children need concrete situations for numbering operations, and from the teacher the language which will help them to reflect upon and define these operations. Out of many different situations they will in time make the acts of generalization which free the numbers from the concrete taking them to the level of abstractions, and in the next chapter we shall examine more carefully what is involved in this very important development.

CHAPTER SIX

FROM THE CONCRETE TO THE ABSTRACT

THE word concrete is derived from the Latin *concretus* (past participle of CON—with+CRESCERE, *cretum*—grow) and in this chapter will be used to refer to the grown-togetherness of the real physical world which is the basis of all human experience. The word abstraction is derived from the Latin *abstractus* (past participle of ABS—away+TRAHERE—draw), and we shall use the term to refer to the process by which we draw away from the elements in our experience to see what they have in common. It involves moving away from the real and actual to the generic, i.e. the formation of classes with defining attributes. An abstraction has been made when the defining attribute of a class has been realized, irrespective of whether or not the concept has been consciously formulated.

An example will serve to illustrate the usefulness of this activity. Let us imagine ourselves going into a room where there is a child at play with his bricks. Almost immediately after being confronted with the whole complex of stimulus factors such as would be presented to us on entering the room, we catalogue what we perceive, using a ready-made set of concepts, thus bringing some sort of order into our experience. The concept 'child' indicates the pattern we are perceiving and also relates this present impression to an indefinite class of similar patterns in the past. Thus to identify what we perceive as 'child' partly depends on being able to use this particular word but also on the fact that we tend to make a consistent response to this particular group of stimuli. Even if we had no words we could still learn to respond differently to the child and the bricks, and in fact little children learn to discriminate and recognize certain objects which have become familiar through experience some time before they learn to speak. The little child whom we have been watching may well show a different kind of response to his teddy-bear from the one he uses to his bricks, because he has discriminated between them.

Many simple class concepts are formed on the basis of the discrimination of certain easily definable attributes. They are the result of the nervous system's response to sensory data—a response which is at once selective and organizing, and which links present with past experiences of a similar nature. Concept-formation occurs because of a readiness in the individual to respond in certain ways rather than others. It involves some degree of generality of application. When the child is presented with some new bricks he generalizes from what is familiar in order to categorize the new, unidentified objects. Moreover, he learns to differentiate among the cases to which the concept applies. Bricks may vary in shape, size, colour and texture and if he is able to use them all for similar activities, it is because he has realized the attribute they have in common in spite of their markedly different properties. But he will also learn to sort his bricks into a number of different classes—all the red ones, all the square ones, all the solid ones, etc. His bricks are now no longer tied to one particular concept, but can be placed in a number of different classes because of his ability to categorize in different ways.

The class concept is only one of many varieties of concept which man can formulate but since it is this kind of concept which is the basis of the language of numbers, we shall confine our attention to these. As we have seen they are the product of a classifying or categorizing mode of response to experience without which the achievements of science and technology would have been impossible.

Class concepts are not always consciously formulated but their presence can be inferred from the habitual and consistent response which an individual makes to a particular type of stimulus situation. For example, after only a few weeks of life a baby will stop crying and smile when the familiar form of his mother comes into view and a little later the same infant will select his rattle from the whole stimulus field around him, pick it up and shake it. Later still he will look for it when it is hidden from sight. In a series of brilliant studies Piaget has shown how the little child comes to form notions of concrete permanent objects as a result of his perceptual response to the real world. These are the constancies in his experience—the people who care for him, the toys he plays with, the blanket which covers him in his cot, the family pet, etc. Once formed these concrete concepts enable him to respond consistently and discriminatingly to familiar objects. They are his first abstractions from the grown-togetherness of the real world.

The basis of a concept is thus a response to experience of a particular kind. It involves a realization of what is familiar in the present event which makes it identifiable, for example, as mother or rattle, a process which involves a withdrawal from the event to realize what it has in common with past events. A concept may therefore be defined as a system of learned responses, the purpose of which is to organize and interpret data provided by sensory perception. This is a psychological definition which would not satisfy the philosopher but will be useful as a working definition for the discussion of concept attainment with which we are concerned in this chapter.

Concepts provide a means of applying past learning to present situations, systems to which we can refer new experience and which help us to bring some order into the flux of sensory stimuli which comes to us through the response of our senses to the real world. This power enables us to accommodate ourselves to our environment and to store knowledge about it, and early in his history man learned to invent words and symbols to stand for the concepts he was organizing. These facilitate the organization of concepts themselves into more complex systems, e.g. classes of classes, which can influence behaviour quite independently of sensory stimulation. This is the kind of conceptual thinking which is going on when we use the language of mathematics.

Like the little child referred to earlier the mathematician makes certain habitual responses to experience of a particular kind. Thousands of years ago he withdrew for attention those variables in the real world which we call magnitude and multitude, and in course of the centuries he has learned to systematize this experience through a vast number of concepts—classes and classes of classes, each of which has a name and a symbol, each of which can be defined by its attributes.

The concepts of the mathematician may seem to be far removed from the concrete object concepts of little children, but they are due to the same essential activity—the activity of abstraction. However, they differ from the concrete object concepts both in their nature and function, and in the process through which they are attained. We must try to appreciate this if we are to understand the kind of learning which must go on if the child is to attain them.

Einstein has always insisted that the concept of number is a free creation of thought, a self-created tool which simplifies the organization of certain sensory experience. Cassirer also held this view, pointing out that number is a purely functional concept. We cannot extract it from the quality of things given through our senses. It is abstract only by virtue of its independence of all particular cases. Numbers are not descriptions or images of concrete things, of physical objects. Rather they express very simple relations. Number is always abstract and must be *thought into* the concrete situation.

In other words the class terms which constitute the language of number were created by men when they found it necessary to categorize the variation in experience which we call quantity. As we noticed in chapter 2 primitive peoples were aware of differences of this kind but they used numbering words with a qualifying rather than a quantifying intention, and because of this they were always tied to particular situations and never available for general use. They had a set of words which they could use to say something about the size of a collection of canoes, but needed another set to say something about a collection of arrows. In contrast mathematical numbers are universally applicable, and progress from the one form of classification to the other involves withdrawal from the concrete and perceptual. The early concrete number forms were perceptually organized units; numbers are organizations of thought, symbolic responses to experience, a product of the fertile imagination of man which once created provided a formula of classes, a basis for classifying experience in a particular way through counting and measuring. The child can only enter into this heritage when he is mentally ready for this level of thinking and some will never make it unaided. This does not mean that he will not pick up by imitation and use some of the words and symbols in the language at quite an early age, but if one examines his responses to quantity as Piaget has done it is clear that the concept of number is absent in his early years. In Piaget's view this is because the mental structures with which he is operating in these early years do not allow him to operate at this level. Nevertheless these primary mental structures are important to the rest of the story because they constitute the basic systems of mental operation out of which in due time mental structures capable of higher-order operations emerge.

From this discussion it should be clear that learning a language is not simply a matter of learning words and symbols, it is a matter of correctly relating words and symbols to the things, events, or ideas for which they stand in our culture. It provides an initiation into the modes of classification used by the groups and involves learning to organize experience in the way our forefathers have done. Language and thinking are intimately connected and the meaning intended by a speaker depends on the kind of classifications and abstractions he has made. Communication of meaning through language cannot take place between teacher and child unless the latter is capable of making the same kind of abstractions which his teacher has made. Cultural influences will determine to some extent the way he learns to classify his experience because through them he comes into the possession of a certain set of classifying terms, and we have already discussed the implications of this fact at some length in chapter 5. What must now be stressed is that the child's modes of thinking and organizing experience will ultimately be determined by the way in which his nervous system operates, the use it makes of sensory information relayed to the brain including the verbal cues presented by his culture. There are significant differences between individuals in this respect, some of which are related to variations in inherited capacities and some to the fact that these capacities are in process of development through childhood. This means that an individual might fail to attain higher order concepts due to poor inheritance, or because at any age his capacity to organize his thinking in this way is not sufficiently mature, or because an impoverished cultural background has left him without the necessary language tools. The fact that a child picks up by imitation and uses class terms like 'blue' and 'five' is no guarantee that he understands the defining attributes of these classes or could formulate the concepts in the same terms as we do.

It should be clear also that to teach a child to say the words in the number series is not the same thing as teaching him to make a mathematical count, and teaching him tricks which enable him to find the right symbol to complete equations such as $2 + 4 =$, or $5 - 3 =$, is not the same thing as teaching him the concepts of addition and subtraction. Yet if our aim is that he shall become able to use the language of number then clearly what we need to provide for is experience which will facilitate the development of number concepts.

This means that we need to know as much as we can about the way in which children's thinking powers develop, what we may expect in the way of capacity for generalization and the formation of concepts at different ages and the varieties in modalities in thinking we may expect to find. Piaget's developmental studies are outstanding in this context and range over the whole field of conceptual activity from the forming of the notion of concrete permanent objects in the child's perceptual experience to the organization of concepts of space, time, number, etc. In a series of studies which include 'The Origin of Intelligence', 'The Child's Conception of Reality' and 'The Child's Conception of Space' (early chapters), Piaget analyses the development of the basic object concepts, showing the way in which they develop out of simpler behaviour patterns. This he does with the help of a theoretical model of the developing structure of the actual thought processes. He believes that all thought originates in the interiorization of actions. The child learns to respond to a certain stimulus with certain actions and after a time to work out how he is going to do these actions before acting. He can do this because his actions and their results are registered in the central nervous system and lead to the establishment of groups or systems of operations which Piaget calls mental structures.

Piaget's primary interest for the last forty years has been in the genesis of mental operations, and he is the first psychologist to carry out a systematic study of concept formation from the genetic point of view. In an impressive series of experimental inquiries in various fields of human experience, he examines the notions of little children and the changes in their ideas which accompany the maturation of their mental abilities. The methods he uses are extremely ingenious and imaginative and his hypotheses are original and challenging. He sets out to show how simple mental operations can be ordered and co-ordinated in terms of systems or groups of operations, and how such groups are discernible in the activities of children. Though the children themselves are not aware of these systems their activities are governed by their nature, and Piaget has discovered a way of disclosing how they work.

Piaget suggests that thought activities may be analysed in terms of groups or systems of operations and that these groupings are relational systems which have certain defining properties, three of which we must mention here. The first is *composition* and relates to the fact that any two unit operations can be combined to produce a new unit. The second is *reversibility* and refers to the fact that any two units which have been combined can be separated again. The third is *associativity*, referring to the fact that the same result may be obtained by combining units in different ways. These unit operations, of course, refer to actions which are capable of being internalized, reversed and co-ordinated into systems of thought. The structures or groupings with which the child is working early in his life are of a different kind from those he will be building up and using later, but a sign of the maturity of the particular structures associated with any particular stage in development is that they demonstrate the properties listed above.

In all the varied fields he has explored Piaget finds three types of mental structure corresponding to three stages in development. 'Sensori-motor group structures' are being built up during the first stage. At this stage the child can only perform actions, but at first there are no operations because the child cannot yet internalize its activities. Nevertheless he begins as we have seen to form notions of objects and learns to construct the notion of a permanent object lying beyond the field of vision. This Piaget demonstrates with experiments in which he hid a watch with which he and a child had been playing. At one stage the child will lose interest if it is covered up but later he learns to search for it. This development implies the formation of the behavioural concept of a permanent object, and by the age of two he has many of these which are the result of the organization of his perceptual experience, made possible by the increasingly effective co-ordination of his bodily movements and the co-ordination of sensory data and movement information into sensori-motor group structures. In these structures Piaget finds the beginnings of intelligence since they demonstrate the properties of combination (as when the child acts in a certain way in response to a sensory impression), reversibility (as when he looks for an object which he has seen hidden, going back to the starting point of the event) and associativity (as when he changes his movement response to a known object).

The second stage covers a much longer period during which what Piaget calls 'concrete operation groupment structures' are being built up. Three phases can be distinguished in this period, and since this is the stage in development with which we are chiefly concerned we shall need to go into more detail. The first phase from about two to four years is characterized by what Piaget calls '*pre-operational thought*'. Now actions become internalized because the child can use imagery to imitate an action. This is the beginning of symbolic behaviour and soon the child learns to use language. At first words are tied to actions or needs, and the words he acquires may be applied to all sorts of objects to which the adult would not apply them. Daddy may be applied to any man, bow-wow to many moving creatures. Piaget emphasizes the egocentric attitude which dominates at this stage, making it impossible for the little child to distinguish between inner experiences and external reality. Objects such as teddies, chairs and bricks are treated as living and endowed with intentions and desires. It is only through a gradual process of socialization that the child learns to accommodate to the use of words by adults. Nevertheless thought has begun because the child is now able to use symbols to represent absent objects and past events. But this thinking is pre-operational because the internalized actions are not yet reversible and conservation is only understood at the sensori-motor level.

The next phase which Piaget finds in his children between four and seven years is characterized by what he calls '*intuitive thought*'. The 'concrete operation groupment structures' are slowly becoming organized but thinking is still tied to perceptual factors, and the structures he is working with are rigid and irreversible. The importance of the achievement of reversibility will best be illustrated by one of Piaget's examples which has already been referred to. Suppose that a child at this stage is asked to put out a set of beads corresponding to a model set and suppose also that he is able to do this and to say how many beads are contained in each set and also to agree that the sets are the same. Suppose now that the arrangement of the model set is disturbed so that it now occupies more space than formerly and that he is then asked whether the sets are still the same. The child whose mental structures are still irreversible will say that the set which is now spread out is 'more because it is spread out' or 'more because it is wider'. This is the most usual response of five-year-old children even to a set as small as one containing only four beads, as the writer discovered when trying out this experiment herself. The response is of this kind because the child is unable to appreciate the operation which has just been performed. He makes his judgment on immediate intuitive perception without reference to what has gone before, since he cannot think back through the experience to the original situation. This example also illustrates the difficulty a child has at this stage in reasoning about the whole and its parts at the same time. The concept of a group as a collection of items is not yet organized. Hence his response to a group is spatially and not quantitatively organized.

Later, to use Piaget's own words, 'the operation is no longer immediately absorbed into the intuitive result obtained. It frees itself, as it were, and becomes capable of moving in reverse. Each transformation can be compensated by its inverse, so that any arrangement may give rise to any other, and conversely'. A child at this stage will respond to the question about the disturbed set by saying something like, 'It's still the same because we made them the same', and the tone of the reply will sometimes suggest that the equality is so obvious that the question is absurd. A response of this kind is an indication that the child now has the concept of conservation of a sum. He understands that the numerical value of the set remains invariant despite any other changes which may occur.

One or two further examples will be useful to bring out the significance of this development from the point of view of the attainment of the number concept. These were devised by Piaget and have also been tried out by the writer with English children. The child is shown a glass beaker containing some wooden beads and asked to put the same number of beads in another beaker of the same shape and size. If he has not gone beyond the level of intuitive thinking he will not try to establish the number of beads in the filled beaker but will immediately start putting beads in the empty one, continuing until they reach the same level as those in the model. In other words he makes his judgment on spatial considerations only. In contrast the child who has the group concept will count the beads in the filled beaker first and then put the required number in the empty one.

For similar reasons if a child still at the intuitive level in his thinking is given a collection of toffees and asked to share them with the adult so that they both have the same number he will immediately separate the collection into two parts and may then go to some trouble to make the two collections *look* the same. He is thinking of the wholes and not of the parts, and his judgment and action are based on what Piaget calls a global view.

Similar results were obtained by Piaget when experimenting with continuous quantity. A child is presented with a glass containing lemonade and asked to fill another similar glass with the same amount. This he will do, often taking great pains to ensure that the same level is reached in the second glass, and if asked will say that each glass contains the same amount as the other. If now he is asked to pour the lemonade from the glass he has filled into another which is taller and narrower so that the level of lemonade is higher from the base and then asked whether the two glasses contain the same amounts he will point to the taller glass and say it has 'more because it is higher'. Again there is no appreciation of the concept of conservation of quantity.

When the child's response shows that he has the concept of conservation of a sum this is an indication to Piaget that 'concrete operation groupment structures' are becoming organized. Intuitive thinking consists in the imaginary representation of complex configurations, and until the child can go beyond this the relations he constructs can be neither combined nor reversed. Now, however, since his actions are internalized and give rise to systems of concrete operations, he becomes capable of classification, the inclusion of one class in another, and of seriation.

If a child who is not yet at this stage is presented with a set of Tillich bricks laid out in a random fashion on the floor and asked to arrange them in order he generally arranges them in a nice neat line without any attention to size differences. In contrast the child who has reached the level of *concrete operations* will more often than not arrange them in a staircase, working in a systematic fashion. The child is making use of one of his systems of operations but throughout Piaget insists on the qualification 'concrete' with respect to these systems or mental structures, since the child's thought is still concerned with actual operations carried out on concrete objects. Propositional operations are quite impossible at this stage.

However, the appreciation of relations which these systems of operations involve gives rise to the concept of number. He has learned to establish a one-to-one correspondence involving the conservation of equivalence in spite of qualitative difference, as in the case of matching the two sets of beads. Also he has learned to form simple asymmetrical relations into a system, as in the case of ordering the set of Tillich bricks. What he has now to do is to combine the operations of classification and seriation so that the unit is treated both as an element in a class and also as a member of a series. While he is still at the intuitive level he can either concentrate on the individual elements in his perceptual field and combine them according to equivalent attributes into classes such as beads, or arrange them according to their differences, e.g. size, in asymmetrical relations. What he cannot do is to treat the collection of objects as equivalent and different simultaneously. The number concept has been attained at the concrete level when a collection of objects is regarded both as equivalent and orderable, and numbers can now be used to say something about a collection as whole and about any individual item in the collection. When we count the fingers on one hand and arrive at 5 we have defined the position of each relative to the others and established the size of the collection.

Piaget's children do not attain this concept until round about the age of seven, but round about the same time other operations appear. Concepts of time and space are formed and various aspects of the physical world are given structure through systems of grouped operations which result in the appreciation of invariants and the attainment of concepts of conservation, though these are not attained in all the fields in which they can apply simultaneously because the mental operations are still tied to concrete behaviour and have not yet been formalized into a system of general principles.

Thus Inhelder reports at seven years the child can mentally cancel out the transformation of a ball of plasticine into a long strip thus arriving at conservation of matter, but not until he is about ten years is he capable of reversible reasoning in connection with weight, and it is only at eleven years that he understands conservation of volume.

It is not until early adolescence, according to Piaget, that the concrete groupings become formalized into systems with a new look. The adolescent is able to form hypotheses and experiment with them. He makes assumptions and draws conclusions from them. He is able to make deductions from hypotheses and not simply from concrete facts as in the earlier stages, indicating the presence of what Piaget calls 'formal mental structures', which give rise to the possibility of '*propositional or formal operations*'. Whereas earlier operations involved reasoning about *things* and *events*, the child is now capable of reasoning about these concrete operations themselves, i.e. operating upon operations, and these give rise to crete operation groupment structures' are becoming organized. Intuitive thinking consists in the imaginary representation of complex configurations, and until the child can go beyond this the relations he constructs can be neither combined nor reversed. Now, however, since his actions are internalized and give rise to systems of concrete operations, he becomes capable of classification, the inclusion of one class in another, and of seriation.

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Differences in behaviour in an experimental situation reported by Inhelder highlight the essential differences in these three types of mental structure or systems of operations, and will serve to indicate their significance for the educator. In a series of experiments with falling bodies on an inclined plane she reports that the five to seven year olds were incapable of recording the experiment or reproducing it. They were not interested in the objective phenomena but simply in what they could do to set it in motion. Any interpretation was purely subjective. The seven to eleven year olds adopted a more objective approach, acting experimentally, measuring and comparing, to obtain the desired result, but they were unable to isolate the principle at work. For children over eleven years the whole situation provoked attempts at deduction and verification of hypotheses. It acted as a challenge and the interpretation of facts became an integral part of the intellectual reconstruction.

Commenting on the results Inhelder says,

> These few facts, which are only a sample of a wide investigation of the experimental attitude of the child and the adolescent, seem to indicate that all learning presupposes an interpretation of experimental data. This interpretation of facts or abstractions is, of course, determined by a group of emotional factors which have been made evident by the learning theorists. But it depends also on cognitive factors, such as the intellectual instruments at the child's disposal for the assimilation and interpretation of physical data, and the experimental attitudes (goals for action) which direct the investigation.
>
> During the first phase the child acts through pleasure of setting something in motion rather than to modify the course of an experiment. This attitude conditions a subjective interpretation of the facts. During the second phase the child acts in order to produce a clearly determined, concrete result; this attitude facilitates a more objective interpretation and requires operations of comparison and measurement. During the third phase the mechanical device becomes more and more an occasion for deduction and verification of hypotheses.

The interpretation of facts becomes an integral part of an actual intellectual construction. Thus the forms of learning cannot be isolated from the instruments of knowledge. Both undergo an evolution according to age.[1]

We are all familiar with the fact that emotional attitudes influence learning, but what Inhelder's experiments show is that if a child is not mentally ready to assimilate an experience he will not feel motivated to explore it except at the level of motor activity. This is an important statement for the Infant School teacher.

Some of my readers may have found this account of Piaget's work rather difficult to follow but it is hoped that the attempt to make his findings more accessible will be appreciated by some. Piaget's works are not easy to read either in the original French or in translation, but those who recognize the importance for teachers of his findings may care to read the more detailed account given by Thomson in 'The Psychology of Thinking' published in the Pelican Series, which I have made use of in compiling this chapter.

There is no doubt at all that important implications for educational theory and practice are involved if Piaget's thesis is accepted. He claims to have shown that development proceeds through a series of clearly-defined stages, each of which is characterized by its own particular type of mental operation which is qualitatively distinct from the others. He appears to conceive of a certain uniformity for all individuals in the growth history of these structures or systems of operations, since it is his view that the course of development can be shown to be not only psychologically but logically necessary. This leaves little room for individual differences in growth history, and in fact Piaget[2] has stated fairly recently that though further research may show that the general rate of progress through the stages may be accelerated or retarded by cultural or other factors, he believes the sequence of stages to be invariant because maturation takes this form.

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[1] Inhelder, B., *Discussions on Child Development*, Vol. II.

[2] See Piaget's contributions in *Discussions on Child Development*, Vol. II.

Naturally his stress on development as a matter of the maturation of mental capacities has aroused a good deal of interest in persons interested in the education of young children. Nathan Isaacs has explored the educational implications of his theory very thoroughly and I cannot do better than quote from one of his articles.

> There are certain basic concepts which pervade and largely control the whole structure of our ordinary adult thought. Chief among these are the notions of space and time, reality and causation; number, order, measure, shape and size; motion, speed, force and energy; and the ideas of the fundamental logical relations, like those of whole and part, classes, class-hierarchies and their members, and implication. Most of us, of course, would not be able, and would not feel called upon to try, to formulate these concepts in strict intellectual terms. But they function in all of us, and *function in a highly organized and structured way*, and it is they that provide the *coherent framework of our normal thought-world* through which we order and interpret all the succession of impressions, happenings and experiences which flow in upon us.
>
> Children pick up most of these words, or words connected with them, quite soon and learn to use them in the right situations. This leads us to assume, very naturally, that they have the corresponding ideas, at least in a simple and elementary form. But here, according to Piaget's findings, we are in the main quite wrong. (See examples quoted already in this chapter.)
>
> Thus we end up with a *tremendous contrast, above all in the early years of the child, between the outward show of his use of language and the inward reality of the actual level of organization and cohesion of his thought.* The basic organizing concepts are to all appearance not there yet, and thus have not yet done their work. Hence, beneath the surface, the world of flux to which all Piaget's experiments with children of four to six so eloquently testify. Towards seven to eight, as we have seen, the contrast lessens; the organizing concepts are starting to take shape and their work begins to show. Yet it is only at eleven to fourteen that, in the average child, the contrast really disappears. *The inward psychological reality joins up with the façade* and Piaget's soundings bring out everywhere just the same kind of responses as any of us adults might

give. *The child now lives in the same functional thought world as we do.* (Italics are mine.)

Nathan Isaacs goes on to suggest that the progressive educationist is now confronted by a problem which he puts in these terms:

> If intellectual development is so much an inward affair taking its own time and if the chief organizing concepts do not begin to mature normally till eight to ten, whilst logical reasoning is only the last stage to be reached, at eleven to fourteen, what exactly is it that modern educational methods, applied from the nursery school age onward, can achieve? What in fact, can or should we aim at? Do our progressive methods, in the intellectual field, just produce another kind of façade? Should we let the central processes of intellectual growth look after themselves, as indeed they must do, and concentrate on other aspects of the child's personality, dramatic or aesthetic or social, making sure only that he enjoys those conditions of communal life which, according to Piaget, are essential even for intellectual advance?
>
> These questions, as I have said, seem inevitably to arise if we accept Piaget's picture at its face value and draw what appear to be the logical conclusions from it. But most of us will probably feel strongly that there are facts on the other side which are too real and massive to be dismissed, even if we are obliged to accept the main tenor of Piaget's findings. And we are therefore left with the sense that these must somehow be corrected or supplemented so that those other facts can also be fitted in.[1]

The present writer is in agreement with the ideas expressed in the preceding paragraph as will be clear in the rest of this and later chapters.

At the time of writing a good many research workers in this and other countries are following up the lead Piaget has given and some of the published results seem to indicate that there are factors making for differences in individual development which Piaget's approach does not reveal.

Recent studies of thinking by Bartlett, Bruner, McKellar, Dienes and others have thrown important light on differences between people in the way they handle experience and form concepts. It will not be possible to discuss these in detail here but one variable which Dienes' research discloses is specially important for our particular study. In a recent article he says:

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[1] Isaacs, N., 'Piaget and Progressive Education', article in *Some Aspects of Piaget's Work.*

> We know that some people must formulate their concepts very explicitly and exactly before they can use them, while others can think in terms of the same concepts with equal efficiency but with only a small amount of such explicit formulation. Or again, some people, as Bartlett points out, tend to think of broad concepts in terms of the internal relationships of the constituent concepts, while others build up their concepts following some intuitively-realized requirement or standard. For the former it is more the detail and its intricacies, for the latter the more over-all picture, that counts. These might be called the analytical and the constructive aspects respectively.[1]

Dienes has found that children's early concepts are much more constructively than analytically directed and this, of course, explains what Piaget calls the global approach to quantity which we have already discussed. The results of his experiments on conservation of quantity with children under six or seven suggest, as Dienes says, 'that the overall visual picture is, at this age, a much more powerful determinant of a concept than are the corresponding percepts of its analytical aspects. Notions of quantity at an early stage are linked with what a group of objects looks like, not with exactly how many objects there are in it.' Dienes draws the conclusion that it is only through constructive activities with structured material that the child can be led on to analytical thinking. 'Constructive exercises have to precede analytical insights. It may be that some children will never reach the analytical stage; the great challenge to the teacher is that of devising ways and means of making the whole of mathematics available to such children.'

This suggests that the function of the Infant School teacher will be to provide children with experiences on which they can operate constructively and gradually become aware, through experiences of grouping, comparing, sharing, ordering and the like, of the relationship between things which can be defined in the language of number when the time is ripe to make their intuitive grasp of the situation more explicit.

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[1] Dienes, Z. P., 'The Growth of Mathematical Concepts in Children through Experience', *Educational Research*, Vol. II, No. 1, November 1959.

Another important study is that reported by Wheeler. She found that the children she tested, representing all age groups from five to fifteen, tended to fall into two groups according to their characteristic mode of reasoning.

> The characteristic mode of one batch is analytic or explicit reasoning, proceeding by a succession of logical steps; of the others, an implicit or synthetic apperception of what many psychologists would call Gestalten or configurations, resulting from a kind of comprehensive insight which integrates the constituent aspects almost simultaneously and so enables them to jump to the right conclusions.[1]

A further distinction was found by Wheeler when she analysed the kind of problems which the children found easy and those which proved more difficult. Many modern logicians classify logical processes into a calculus of 'classes', a calculus of 'relations' and a calculus of 'propositions'. Wheeler found that problems of the first type were easiest for her children and refers to Burt's suggestion that the calculus of classes is easier than the calculus of propositions (or statements) because 'the former involves thinking in terms of the "object language" and the latter in terms of the "meta-language"—that is to say thinking which proceeds by making statements about statements, a procedure which is obviously more complex.' As one would expect from Piaget's work, Wheeler found that the differences between the reasoning of young and older children could be defined in these terms, but she adds the very important qualification that 'when occasion requires (e.g. in the case of a dispute between two or more children) even quite a young child will drop into the "meta-language". And the difficulty of the latter mode of reasoning springs, not from some qualitative difference which the younger child is unable to master, but merely from its greater complexity.' And she adds, 'No doubt, the maturation of intelligence with increasing age may be one factor in facilitating the change; but the main influence is the child's free intercourse with other children, particularly when the latter vary in age and understanding.'

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[1] Wheeler, D., 'Studies in the development of reasoning in school children', *Brit. J. Statistical Psychology*, Vol. XI, Part II, November 1958.

Wheeler also found that when a group of children varying in age from seven to thirteen, but all with I.Q.s round about 100, were presented with two different tests, the type of solution used by the children varied from one test to the other, and draws the conclusion that 'the age at which the changes in type of solution appear is related, not so much to formal differences in logical reasoning, but rather to the material nature of the problem'.[1]

Another interesting finding of Wheeler is that

> the method which the child adopts in trying to discover a solution seems to depend quite as much on temperament as on the development of reasoning ability. The boy who tries to think out a reason for the phenomena, and to discover from actual trial whether his own explanation is right, is an inquisitive and not necessarily a logical child. . . . Even the substitution of formalized reasoning for concrete reasoning (or, as Burt would say, of the 'meta-language' for the 'object language'—a change which Piaget regards as a new and distinctive step achieved only towards adolescence) seems due rather to the child's emotional attitude than to his cognitive capacity: it is the aggressive, argumentative child who tends to express his reasoning in the form 'It isn't true that (so-and-so) because, it it were, then you couldn't say (so-and-so).' Much depends on the social context of the inquiry: a young child working by himself is likely to talk or think chiefly in the object language; but when two or more children are wrangling together, then each changes to the meta-language to justify his own conclusions.

Piaget has been criticized for giving the chronological ages of his children rather than mental ages, and the suggestion has been made that qualitative changes of the kind he describes may occur at certain mental ages, but Wheeler found that 'in children of high ability' (e.g. a child of seven with a mental age of ten), 'the development of reasoning not only appears earlier, but also follows a somewhat different course' from that taken by children of average ability.

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[1] Ibid., p. 152.

Clearly Wheeler's findings differ in many important respects from Piaget's, and my own experience, both in observing children at work in school over a number of years and during the course of an experimental study which will be referred to later, leads me into agreement with many of her conclusions with regard to the variations in children's reasoning.

The findings of Piaget, Dienes and Wheeler all suggest to me that the eventual attainment of the number concept depends on the opportunity the child is given to handle real things rather than on the committing to memory of number bonds through frequent exercises with sum cards, and the results of an inquiry reported recently by A. A. Williams[1] lend support to this conclusion. He gave two series of tests to infant-aged children; one series of tests referred to as the Test Battery, was similar in kind to those used by Piaget and designed to reveal the state of understanding regarding the basic concepts involved in number; the other series, the Attainment Test, contained tasks involving enumeration, counting-on, and reversibility. He reports as follows:

> In comparing the results of the Attainment Test with the final assessments on the Test Battery, it is apparent that some children are able to perform computations, with some degree of accuracy, prior to any real appreciation of the nature of number relationships. Such children, on further investigation, are unable to make any but the simplest generalizations, and it must remain a matter of conjecture as to what proportion subsequently become unhappy and insecure in relation to number activities.

He draws the conclusion that

> the part of the teacher is to bear in mind the relational nature of the number system and the developmental nature of the young child's

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[1] Williams, A. A., 'Number Readiness', *Birmingham Univ. Educ. Review*, November 1958.

comprehension of these relationships. Her task is not so much to teach, in the sense of disseminating information, as to provide ways and means of helping the child to think about quantities in accordance with his developing notions.

These findings are in accord with those made by the present writer which have been recorded in detail elsewhere.[1] In this experimental inquiry a group of five-year-old children became engaged with the experimenter in a carefully-designed programme of play activities in which materials were handled and the operations undertaken talked about with the experimenter. These included sorting and grouping, comparing and sharing, ordering and numbering. Discoveries were made and talked about so that the children who were ready were able to abstract the relationships implicit in the situations, and apply their new understanding in other situations. The results suggested that where the children's environment is carefully planned to involve experience of these relationships at a concrete level through play activities and stories which interest them, and where the teacher shares them with the children, helping those who are ready to make explicit those relationships already known implicitly and acted upon, children can be helped in their development towards an appreciation of the meaning of number.

It is important to recognize in this connection that some numbering or measuring experiences are more conducive to the abstraction of these relational concepts than others since the nature of the materials themselves forces attention on the ways in which they are related to each other. The authors of *The Teaching of Mathematics in Primary Schools* express the view that 'through constructive play, experiment and discussion children become aware of relationships and develop mental structures that are mathematical in form and are in fact the only sound basis of mathematical techniques', and that the aim of Primary Teaching 'is the laying of this foundation of mathematical thinking about the numerical and spatial aspects of the objects and activities which children encounter'.[1] While expressing full agreement with these views, the point to be stressed here is that some encounters are more conducive than others to the development of relational thinking; and the teacher needs to be able to assess the possibilities for learning in the experiences she provides.

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[1] Churchill, E. M., 'An Experimental Study of the Growth of Numerical Ideas in Five-Year-Old Children', unpublished thesis, 1956, Leeds University Library.

Some mathematicians, notably Cuisenaire and Catharine Stern, have taken the view that it is important for children to handle materials which represent in concrete terms the relational characteristics of the number system, believing that through experimental play, and more directed work with them, children will be helped to appreciate the structure of the system from the very beginning, abstracting the relations which it expresses.

Stern believes that 'to treat arithmetic as a compound of disconnected number facts is a direct violation of the inner nature of numbers', and says of her approach that it is 'based on the assumption that to learn arithmetic means to grasp the fundamental relations from which the single number facts derive sense and meaning'.[1] She points out that the counting of collections does not foster relational thinking and for this reason in her structural material,

> instead of developing the first number concepts by *counting* the elements of unstructured groups of objects, the child works with clear structures that show him from the start the relations between the numbers of our number system . . . the child's experiments with our materials are based upon *measurement*, which introduces the basic mathematical concepts at the very start of the child's number work.

As Stern herself points out 'the use of blocks in the teaching of arithmetic is by no means new'. They have been used in many methods from the time of Pestalozzi on, notably in the

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[1] Maths. Assoc., *The Teaching of Mathematics in Primary Schools.*

[2] Stern, C., *Children discover Arithmetic.*

Tillich number kit and in the Montessori system. In these methods, however, even though the 'two addends are given concrete form, their sum has to be found by counting the units one by one, and the inter-relationship between numbers is hardly seen'. With the Stern material, because the child is measuring instead of counting, the relationships can be seen and discovered by the child himself in the course of experimental play. This will be clearer after reading chapter 8.

Though Stern offers a wider variety of structures in her set of materials than Cuisenaire, their approach has this in common that both provide visual models of our number system with its base of 10. Both, in common with Madame Montessori, see this as appropriate and valuable experience for infant-aged children, though it is obvious from their writings that they would expect this to involve a good deal of directed activity by teachers. In my very limited experience of this kind of material in infant classes I have found that little children enjoy making patterns and building with the blocks when they are first introduced into the classroom, but unless there is a good deal of direction from the teacher they do not achieve their purpose as tools of number education. Moreover, because the range of shapes and sizes is limited their possibilities for constructive activities are exhausted pretty quickly.

If Piaget's finding that most children are unable to understand what numbers stand for in our language until round about seven is found to be true of English children, then their observations are not surprising and it might be thought that structural material of this kind would be more useful in Junior schools or with older backward children. However, until more sustained experimental work has been undertaken and reported upon, it would be premature to form conclusions about their value. What is important is for teachers to keep themselves informed about current research concerned with the development of children's powers of perception, thinking and reasoning; and consider very carefully what it suggests as constituting appropriate learning experiences at different ages.

In contrast to Stern and Cuisenaire, Dienes has the needs of junior-aged children particularly in mind in designing his material, though he considers that it may be found to be a useful addition to the play materials provided for younger children since through constructive operations with, for example, the multi-based blocks, they would be likely to become aware of some of the relationships as they used them selectively for their own purposes. Dienes starts from the premise that three types of concept (pure mathematical, notational, and applied concepts) have to be acquired before the child can use the system intelligently in mathematical operations. 'Pure mathematical concepts are those that deal with the number alone, or with relationships between numbers, and are independent on the ways in which we express numbers. . . . They express properties of numbers as numbers and not properties of the way we express numbers.'[1] These include the number facts we usually require children to commit to memory in the Junior school, but many other facts besides. 'Notational concepts are those that deal directly with those properties of numbers that arise as a direct consequence of the ways in which we express numbers.' For example, in 204 the first digit means 2 hundreds, the 0 means the absence of any tens, and the third 4 units. As Dienes points out certain pure concepts form the basis of the notational concepts. For example, the concept of grouping in certain size groups is a pure concept, but the concept of grouping in tens and the accompanying convention of place value is notational. This suggests that the pure concept must be developed before the notational concept can be attained, since the latter is based on the former. Yet in the past we seem to have relied upon practice with sum cards to yield the pure as well as the notational concepts, if indeed we thought about concept-formation at all. 'Applied concepts are those which express different facets of the real world as we find it by means of already formed mathematical concepts.' These include concepts of length, weight, money-value, etc.

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[1] Ibid.

Dienes agrees with Stern and Cuisenaire about the value of experience with specially structural material to aid the development of these concepts, but he recognizes that many of the concepts which children form as a result of their everyday experiences lay foundations for the concept of number. Every experience has its own structure and invites the identification of separate objects and later the recognition of similarities between objects and the grouping of these into classes. Responses to experiences of this kind may include 'counting', though when this first occurs, as we have seen, it is not a mathematical count, but rather a verbal play, an imitation of an adult activity. All these experiences are foundations for the concept of number and at some point there occurs what looks like a sudden insight—a moment of realization—when the constituent parts of the number concept become connected and remain so for future use. Naturally the good teacher will be aware of the learning possibilities in these real life situations and exploit them to the full, but if the nature of the higher order number concepts is examined it will be realized that the number of real life situations that can usefully lead up to them diminishes as they become more complex, and in Dienes' view, 'if we still wish the child to form concepts as a result of experience, then we must provide him with artificial experiences to lead up to these more complex concepts and processes. In other words, there is a serious gap in a number of fundamental cycles (of concept formation), and it is the specifically educational task to fill in these gaps. . . . If we are agreed on the necessity of providing such experiences to help mathematical learning, then we are faced with the task of constructing a series of pre-conceptual experiences which will fit into the children's natural concept formation cycles and will generate the required insights in at least the majority of cases'. Dienes sees this as an urgent need at the Junior school stage when the attainment of higher order concepts is necessary to an intelligent approach to the operations the child is required to perform.

Since in this book we are chiefly concerned with the learning of younger children we shall not examine his material in detail but I wish to draw attention to one fundamental difference between his material and that of Stern and Cuisenaire because it is based on a principle which should, I believe, influence the planning of Infant teachers also. This is what Dienes calls the variability principle. As we have seen the attainment of a concept depends upon the ability to identify and define something which a number of different concrete experiences have in common. Before this can happen the experiences must have been internalized in the form of images which provide the perceptual equivalents of the concept. Dienes' hypothesis is that the more varied the perceptual models at the child's disposal, the easier the acquisition of the concept. 'It is when we have had dealings with a variety of perceptual equivalents of a concept that we discover the purely conceptual structure by seeing that there is something in common to all the perceptual equivalents. It is thus that a really mathematical insight is achieved. A visual aid, or any other aid, although no doubt better than no aid, merely links the symbolism of mathematics to a visual equivalent. The "picture" of a concept is not the concept itself. . . . It is the realization of the common properties of a great many different such representations that constitutes mathematical insight. We must pass from the precept to the concept, this being the process of abstraction.'

The story quoted earlier of the little boy who identified 'five' with a particular arrangement of dots had a 'picture of the concept', and the fact that he was unable to identify a group of five ducks because the arrangement was different showed clearly that he had not attained the concept of five. In accordance with the variability principle Dienes provides a number of quite different structures on which mathematically equivalent tasks can be performed. This allows for the formation of a number of perceptual models involving visual, tactile and proprioceptive imagery, and in time the children come to realize the sameness in the tasks performed with different materials. It is with this same principle in mind that in the later chapters of this book many varied experiences are advocated as ways of contributing to the development of the concept of number.

Dienes holds that it is also important to provide for mathematical variability. In this case the structure of the concept is preserved while the variables which go to make up the concept are varied. For example, in designing material structured to facilitate the attainment of the place value concept, instead of using blocks structured to the base 10, as Stern and Cuisenaire do, he varies the base and the powers to which the bases are raised. In what are called the Multibase Arithmetic Blocks there are a number of different sets of pieces, the volumes of which are in ascending geometric progression with different common ratios. The pieces in each set are referred to as units, longs, flats and blocks; and in the set with the base of 4, for example, the long consists of 4 units, the flat of 16, and the block of 256. The children play games with this and other sets with different bases, and as a result gain the 'feel' of the geometric progression which paves the way for an appreciation of the place value concept.

This kind of material has been found to be of value with Juniors but the principle of variability is surely one which should be borne in mind in planning activities for Infants. To replace the traditional sum cards with cards on which collections of dots occur in place of the numerals, or by exercises with a narrow range of structural material seems to me of very doubtful value. At the same time I think that some children reach a stage in their development while still in the Infant school, when their capacity for further learning is held up because teachers are not providing the kind of experiences which contain within them the possibilities for structuring in ways which promote the attainment of the necessary concepts. In saying this I am not thinking only of specially designed structured material, most of which I am inclined to think is more appropriate for Juniors, but of the material we provide for play activities such as bricks, jig-saws, skittles, etc.; the stories we tell to children and act out together; the aids to counting we provide, and the use we make of the opportunities which daily life experiences offer for counting, grouping, sharing, comparing, serializing, etc.

In writing about his material Dienes makes it clear that what he has particularly in mind are the needs of children at Piaget's stage two in respect of any concept. This is the intermediate stage when imagery plays such an important part. At stage one, as we have noticed, the child interprets a problem situation in terms of certain dominant perceptual qualities, e.g. the space occupied by a group of beads; at stage three he has got hold of certain relevant ideas, functional principles on which he can now base his interpretations, e.g. that the quantitative value of the group is conserved in spite of any alterations made in the distribution of its elements. At stage one he brings to bear upon the problem such anticipatory schema, or mental models, as seem to him relevant, and structures the situation in terms of these schema. These may be inadequate to deal with the situation and result in the kind of misinterpretations we have described. By stage three it is evident that these schema have undergone considerable restructuring to yield new schema which make possible a true appreciation of the relations involved.

What has gone on between these two stages is a phase of anticipatory activity, reconstituting of schema, and the building in of new connections. In all this imagery plays a major part. The child can image before he can structure, and it is through his play with images that he is able by intuition to evolve new operational structures. But images are representations of real experiences, and depend for their existence on an initial concrete experience and the capacity to retain and recall it at will. The child who is deprived of experience at the concrete level which is appropriate to the kind of learning desired is deprived of the sources of the images on which his capacity to learn depends. These images are not ideas, they are the stuff on which imagination works in such a way that ideas are born from the activity. We know very little as yet about the help we can give the child at this stage, but it seems likely that if one is thinking particularly of learning which will lead to the attainment of mathematical concepts, structured experiences which facilitate the formation and use of a particular set of images might be of considerable assistance to the child at this stage. But, of course, he does not only need to form images of physical things, but images of his operations upon them, e.g. what happens when he changes the situation in some way, and what it feels like. This means that he must be allowed not only to see materials and watch other people operating with them, but also to operate on them himself. Gradually through this kind of experience and the images he forms of it he comes into possession of mental models which enable him to make predictions in imagination about what will happen if situations change, and through this activity builds up connections, discovers relationships, reversibility, and the constancies in situations.

There is still much to be learnt about the psychological links between perceptual experience and imagery on the one hand, and images and thinking on the other. We know that wide variations between individuals can occur. These include differences with respect to vividness of imagery and preferred modalities, also with respect to the use people make of images in thinking. The studies undertaken by McKellar[1] are extremely interesting and important in this connection and his book can be warmly recommended to all teachers, since the variations he describes in adults will also be found in children, and are educationally significant. For some children proprioceptive imagery is likely to play a much more important role than in others whose visual imagery is particularly vivid. Similarly some children think with verbal and other symbols much earlier than others, while some will tend to rely heavily on sensory images throughout their lives. These facts provide another argument for the variability principle, since the more we vary the experience we offer to a group of children the greater the likelihood that we provide a basis for fruitful learning for each individual.

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[1] McKellar, P., *Imagination and Thinking*.

One thing which McKellar stresses is that all image-making depends on a basis of perceptual experience. This applies as much to the imagination images we use in attempting to share the experience of a teacher or writer who is describing something to us, e.g. a visit to Westminster Abbey, as to the memory images we can evoke in recalling a visit to the Abbey we have ourselves made. It will be impossible for us to share his experience at all unless we have visited the Abbey at some time or some relevant primary, or first hand, experience, e.g. of other churches, with which to associate it.

Perception, whether primary or secondary, provides the raw material out of which thought products are formed, and the power to image enables us to work over perceptual impressions and organize them, relating the impressions derived from one experience with those of others. Imagination, however fantastic and apparently removed from reality, never consists in inventing something out of nothing, but in bringing together elements hitherto unconnected. The relating process is affected by perception itself. As McKellar reminds us, Spearman held that creative thinking consists in perceiving a relation between two events A and B, and then through observation conceiving the possibility that the same relation will hold between X and Y. Analogies, whether implicit (as in metaphor) or explicit (as in simili) are the result of this kind of activity, and the frequency with which we find it necessary to use these in thinking and in communication suggests how vital to us is our experience of the real world. As McKellar says, 'We may regard this use of implicit or explicit analogy from perceived objects and events as a sort of core round which the thinker models his thoughts and communications . . . original thought always depends, both for its form and for its content, on past perceptions, and quite often on past perceptions of a very concrete kind.'

McKellar uses the term 'mental models' to denote the more or less explicit figures of speech which are used to assist thought, understanding and exposition. He points out that they always 'refer to concrete objects, situations, or other areas of knowledge with which a thinker assumes his audience will be acquainted', and quotes examples to show how dependent upon them scientific thinkers are as well as artists. Thus Grey Walter, writing on brain functioning, finds the analogy of the totalisator more adequate to the events as he understands them, than the model of the telephone system favoured by earlier workers in this field. Psychological writings are, of course, full of concepts such as adjustment, derived from the field of mechanics. These two examples illustrate the fact that the models we choose are selected from our own personal experience, and limited by it. They are therefore greatly influenced by the kind of society in which we live and the kind of education we have had. The modern thinker has access to complex mechanical, electrical, and electronic models, and his analogies have outgrown those used by his Victorian counterpart. It must not be overlooked, however, that while modern man has a much larger supply of models at his disposal with which to enrich his imaginative activity, the danger of false analogies being used is thereby increased, a point discussed by McKellar.

This discussion could be developed in several important and interesting directions, but I must content myself with drawing attention to its relevance to educational practices. Teachers, like everyone else, make use of analogies in their communications; unless the children have the necessary experience which makes it possible for them to understand the analogy the communication fails. Children, as well as adults, need the kind of experience on which they can build fruitful models to use in their thinking. Like adults, their models are derived from their direct experience of the real world at home and at school, and in these days is enormously enriched by what McKellar calls 'secondary perceptual experiences' offered by T.V., the Cinema, books, etc. Teachers should examine constantly the experiences their children are having outside school, and those they are providing in the classroom, from the point of view of their model-building possibilities. It is their responsibility to see that useful experience is fed into the imagination, and in terms of number education this means experience which can be structured in particular ways to facilitate the formation of useful mental models.

We noticed in an earlier chapter how, as a result of his sensori-motor and affective responses to the world around him, the little child builds a store of operational knowledge which he uses as a kind of frame of reference, a map of expectancies to guide him in his approach to present experiences which he can call on at will because of his power to image. What we are now concerned with is the conversion of operational knowledge into conceptualized understanding, through the process of abstraction. The teacher is enormously important at this stage in the child's development. In the past educators considered that verbal instruction and repetition were the only means of instilling into the child the knowledge he must acquire. We know better now and if, as the Hadow report bids us, we think in terms of 'activity and experience, rather than of knowledge to be acquired and facts to be stored' we shall be nearer to the picture presented by modern child psychology.

The child learns through his own explorations, experiments and discoveries. The teacher's role is to provide an exciting environment which evokes curiosity, impels response, and extends experience. She can help him to look, to listen, to feel more acutely; to reflect upon the meaning of the experiences he is having; to discover and define relationships as she shares them with him. In this way she is feeding his imaginative life and easing him towards increasing levels of abstraction as he becomes capable of them.

In subsequent chapters we shall be concerned to examine the development of number understanding and skill in these terms.

CHAPTER SEVEN

FROM COMPARISON TO MEASUREMENT

As the importance of 'operational knowledge' has already been stressed we might begin this discussion by suggesting the kind of knowledge children acquire while operating on their toys. Some are heavy, others light; some hard, others soft; some long, others short; some are cornered, others rounded; some are thick, others thin; some rough, others smooth. Now many children arrive at school for the first time able to use words such as these and to show by their usage that they represent abstractions of qualitative differences in their experience. How have they come to this understanding? Can one suggest any other way than through the information conveyed to the brain as they handled the toys in question. What is the difference between heavy and light except the difference in the amount of effort needed to lift or push objects about in space?—information conveyed in terms of proprioceptive sensation. What is the difference between hardness and softness other than the resistance met when handling various objects?—again proprioceptive information. What do corners and roundness mean except something experienced as tactile and proprioceptive sensation when surfaces are explored. Similarly roughness and smoothness are discovered in terms of resistance to touch, again a combination of tactile and proprioceptive information. Long and short are relations distinguishable by sight and also in terms of spread of arms required to lift and move, to get from one end to the other.

It seems clear that differences in experience of this kind are noticed and classified fairly early in experience and many of the motor skills shown by children newly-arrived at school imply a good deal of operational knowledge of this kind.

Furthermore, many children show an interest in the behaviour of materials. They will spend a good deal of time handling clay, rolling it out into long snakes, patting it out into flat pancakes, rolling it into balls. Sand gets handled in the same way to make little castles, to fill buckets, to pour through the hands, to make holes and tunnels. Balls are rolled, thrown, squeezed, kicked and sat upon. Bricks are arranged one behind the other in a line and pushed along as a train, or built up one on top of the other to make a tower. The flannel is filled with water, squeezed out, filled again, squeezed out and eventually hung up to dry. The shiny surface of the doll is thoroughly washed and dried and then carefully wrapped round with a soft blanket and put to bed. The climbing frame is something you walk up and then slide down the other side, but as soon as you feel safe doing this you experiment with other ways of going down the slide—head first, feet first, on your tummy, on your back, walking down, running down, jumping down.

Different qualities in experience become important in all this activity. They are noticed and registered for later reference and, when words are available, become abstracted.

The following examples are typical of responses at this stage and could probably be matched by many readers who are teachers of young children. Two-year-old Jim had been watching his uncle and cousin who were visiting the family. Suddenly he pointed to his uncle and said 'Big', and a few minutes later, jabbing his finger into the face of his three-year-old cousin, he said 'Little'. A little two-year-old girl in a factory nursery put her handbag against the teacher's bigger one and said, 'I've got a bag too, mine's the little one.' Some four-year-olds were in the Wendy House with their teacher. Suddenly Alan put his hand against hers and said, 'Mine's much smaller than yours,' and then, measuring off the distance with his fingers, 'by that much.' Then he took Flo's hand

and measured it against his own, showing that hers was smaller still. Four-year-old Derek, building a tower, said, 'I am going to make my tower taller than you are.' And another four-year-old, Brian, confided, 'Mine's going to be a long, low garage for my cars.' Sheila, wanting some help asked, 'Will you reach that down? it's too high for me. You're tall and can get it.' Pamela, aged five, was interested in heights. She said, 'Richard is bigger than me and Diane is smaller, so I must come in the middle.' Another five-year-old, Joyce, brought some flowers to school. She went to collect a vase for them and brought it to the table where she had laid the flowers down. Then, on looking at the flowers she said, 'That won't do' and went away to return later with a taller vase. These are differences which could all be defined more exactly in terms of linear measure but at this stage no-one is bothered with the question 'How long? How tall? How big? How small?'

Before examining the development which leads to this stage we might notice other responses at the same level. Stephen (4½) was astonished to find that some things floated on the water in the tank and others sank. Susan (4) was overheard explaining to him, 'It's because one's heavy and the other's light.' Stephen (5½) to his friend Norah, 'You'd better use this smaller hammer and let me use the big one, it's heavier than that one.' Two five-year-olds were comparing their suitcases. Marilyn: 'My case is much lighter than yours.' Pamela: 'Do you mean a lighter colour, or not as heavy?' Marilyn: 'I mean it doesn't weigh as much.' Pamela: 'But you can't tell until you hold them both together.' Later on these children will want to ask, 'How heavy?' 'How much heavier?' but not yet.

A little girl was playing in her father's garden. She lifted a large bucket and tried to put it into her small painted seaside one. After a moment of apparent puzzlement, she laughed, dropped her own bucket into the larger one and ran to show her parents. Five-year-old Denise seemed puzzled when she found there was too much water in a large tin to go into a smaller one. Suddenly she realized why and exclaimed, 'This tin is too big.' Christine expressed a similar discovery rather differently, 'This is smaller than the tin so the water won't stay in.' One little five-year-old joined two thistle funnels together with a piece of rubber tubing, and filled them both with water. When she lifted one above the other she found that the water from the higher one ran down into the lower. As she remarked to her companion, 'Water won't stay up if it can get out at the bottom.' How much? is not a question of interest at this stage.

These examples are typical of the interest children show in spatial relations and relations of magnitude. Clearly some of these suggest the possession of 'usable ideas', and very useful they prove to be in the development of building skills, imaginative play with water, clay, sand, etc. This level of abstraction serves to satisfy the child's needs for quite a long time and the idea of asking, 'How much?' 'How long?' 'How heavy?' does not seem to arise.

Very often it arises first in the context of imitative play. Perhaps father has been using a ruler while doing a job of carpentry in the house, or mother a tape measure while making a dress. Perhaps the children's height is being measured and weight recorded in connection with a medical inspection. Incidents such as these will be watched with interest and imitated in play, and if foot rulers, yard sticks and tape measures are available to the children for this kind of play rudimentary matching operations may begin.

Before discussing possible developments at this stage it will be wise to consider what we are doing when we measure. This question will first be considered in the context of linear measurement; weight and capacity measures being considered later in the chapter. If the question is asked about something, 'How long is it?' there is no other way of answering the question except by saying, 'As long as . . .' What we do in fact is to find something to match it against. The Egyptians used various parts of their body to match against: the distance from elbow to the tip of the longest finger (cubit), the arm span, the width of the middle finger (digit), the width of the palm of the hand, the width of the outstretched hand, the length of the foot, etc. Since the proportions of men are very similar this gave the first table of linear measures:

4 digits make 1 palm
3 palms ,, 1 span
2 spans ,, 1 cubit
4 cubits ,, the height or stature of man.

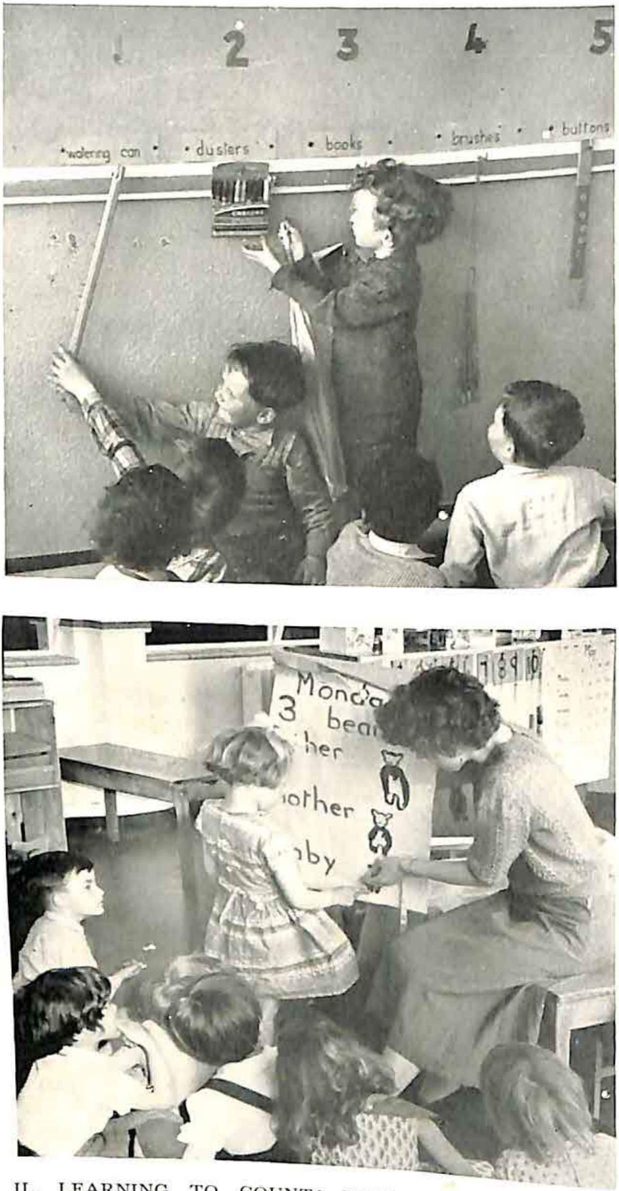
The story of the development of linear measures culminating in the tables of standard measures now in use is an interesting and instructive one. It need not be told here since the story of this and other measurements has been well told by Thyra Smith in a delightful series of booklets entitled *The Story of Measurement*, booklets which will be enjoyed by children as well as teachers.

What we do when we measure length or height then is to match the object in question against one of our standard units which enables us to say in effect 'As long as *x* inches, *x* yards, or *x* feet.'

At first for the child a rough approximation is felt to be adequate as was the rough measurement by body units. The development of standard measures arose when there was a need for greater accuracy. But even now it is important to recognize that measurement can only be as accurate as the tool used allows, and there are many occasions in everyday life when extreme accuracy is unnecessary. The draper who measures off 3 yards of material for a customer only takes extreme care when the material is very expensive and usually the customer comes away with 3 yards and a bit more. The architect cannot be content with approximate measures of this kind, however, and uses tools which enable him to measure down to fractions of an inch.

The following examples illustrate the kind of development teachers may expect if they give children measuring tools of different kinds to use in their play. Janet (5) had bought a piece of ribbon from the shop which was 24 inches long and

I. EXPLORING SHAPES AND SPACES: THE MIRROR; FITTING TOYS; A CLIMBING FRAME.



II. LEARNING TO COUNT: MATCHING COLLECTIONS TO NUMERALS; MEASURING BABY BEAR.

Christine had a piece 12 inches long. With these they began 'measuring' the bookcase, a chair and several other things in the classroom. After a time Janet remarked that her ribbon was longer than Christine's so of course Christine could never measure as much as she could. Stephen and Maureen (both 5) were measuring things with their hands and getting different answers. Stephen explained to the teacher, 'Maureen's and mine are both right,' and when she asked how this could be he said with much giggling, 'Well, she's only got little hands and mine are big.' Alan (5) was working at the woodwork table. Presently he came to his teacher with a puzzled expression on his face and a piece of wood in his hand asked how wide it was. They got a ruler and found it was 1 inch wide and then the teacher asked him why he wanted to know. He explained that he had taken a nail from a tin marked '1½ inch nails' and when he knocked it into the wood it disappeared. 'Why didn't it come out at the other side?' he asked.

The following children were all six years old. One little boy was busily measuring with a footrule when he discovered that the top of his desk measured 3 feet and some over. He said, 'There seems to be half a foot left over, Miss.' Then he measured this part again and counted the inches on his ruler and said, 'Well, six inches must be half of a foot.' Pauline and Angela approached their teacher together and Pauline said, 'The door is three of my footsticks wide.' Then Angela added, 'But it's only one of my yardsticks.' They went away discussing this finding and later came back to say, 'We've measured it and three of the footsticks are the same as one yard.' Dorothy came up to her teacher one day clutching three rulers and asked, 'Will you come and hold these for me? I can't do it on my own.' 'It' proved to be finding the answer to 'How wide is the gate?' After a short discussion of the difficulty Dorothy said, 'I could use one of the long rulers', and went off to fetch one. Later in the morning she was seen approaching John who was facing the same problem, armed with her yard stick. Pauline had measured the length of the teacher's table. She came to say, 'It's forty inches but I shan't write it like that. I'm going to put 3 feet 4 inches because that's the same thing.'

These examples seem to offer strong evidence in support of the claim often made that small children are able to learn through their own activity and experience. It seems likely that the discoveries they make in this way will be more surely understood and more firmly rooted than facts conveyed through verbal instruction even when accompanied by visual aids. The importance of the teacher is also fairly obvious. These children seem to take for granted that she will be interested in their discoveries and willing to help when asked. She has provided a variety of tools to measure which might include, as well as those mentioned, tape-measures, a trundle wheel, a pair of calipers, and a six-foot measure on the wall against which the children can measure each other's height. It has been suggested that an interest in measuring may arise spontaneously following something which the children have seen happening at home. If this does not happen it will, of course, be part of the teacher's job to start the interest going herself.

Some children will have exhausted their interest in measuring things just for the enjoyment of measuring before they reach the end of their Infant School life. Measuring will now only be of interest as it is useful in the service of something else such as making clothes for a doll, following a knitting paper, making a model according to instructions. Such children should be noticed by the teacher and given tasks which are a challenge to their greater ability and skill.

Turning now to the measurement of weight we must first notice that the matching operation this time is of a rather different kind. It is achieved by balancing the thing we wish to weigh against something else, and the answer to the question 'How heavy?' is equivalent to saying 'It's as heavy as one, two or three of these' as the case may be, 'these' being in our culture, ounce and pound weights, etc.

The first balancing operations were done in the hands and then in scale pans held in the hands. Now we have scales of many different kinds in use, some of which are much easier for the little child to understand than others. The simplest form consists of two pans in which the things to be weighed can be put in one pan and balanced against a weight in the other. This kind of balance is fascinating to children when they meet it for the first time and they need time to enjoy its up and down movements before they will be ready to use it for weighing. At this stage a few tins containing such things as stones, beans, shells, peas, will provide the basic experience which leads to an understanding of matching with weights. Later they will be interested to weigh out quantities to match the standard weights available.

Again some examples of six-year-old children will serve to illustrate the kind of developments to be expected. Margaret said to her teacher, 'You need a lot more peas to make 4 oz. than sweets.' When asked why she replied, 'The peas are a lot lighter than the dolly mixtures so you have to put more on the pan.' Philip came rushing up to the teacher excitedly and said, 'Do you know that the 4 oz. and 1 oz. weights weigh the same as the 2 oz. and 3 oz.?' Sheila was weighing out packets of tea in the shop using an 8 oz. weight. She began by emptying nearly all the available tea into the pan to balance the 8 oz. weight. When she had done this she proceeded to fill a tea packet and then found she had enough to fill another as well. Alan, who was looking on said, 'They must be 4 oz. packets' So they weighed them to make sure. Graham said to his companion, 'You don't need as many beans as you do peas. I expect it's 'cos they're bigger. I've noticed that with things that are bigger.' 'Yes,' replied Linda, 'it's because the biggest things are heaviest.' What has Linda still to learn?

The measurement of liquid quantity involves yet another kind of matching operation. This time the answer to 'How much?' is given in terms of spoonfuls, cupfuls, jugfuls, etc., and later in terms of measures which hold a pint, a quart or a gallon. Five-year-old Peter was greatly attracted by the tray of silver sand and at first played experimentally with it, ignoring the other children. After a week of this he began using the containers for counting and measuring. One day he had a large container and a small one. He filled both with sand, then held them up high and emptied them, remarking that he must find another tin like the large one so that they would be empty at the same time. He experimented with various tins till he found what he was looking for. On another occasion Peter was seen counting how many teaspoons full of sand it took to fill a small container. This was 'petrol' which was needed to fill the cars. Margaret (5) was playing with her doll in the Wendy House. It had mumps and she went to the water-trough to fill a large medicine bottle with water and returned to the house. A few minutes later she returned to the water to empty some of the contents of her bottle into a smaller one, explaining, 'The doctor says the other was too large, baby isn't so poorly and only wants a small bottle of medicine.' Michael (5½) was playing at the water trough with Brian. 'Do you know it took nine of these little tins to fill the bowl with water, but if I use this bigger one it only takes four and a bit to fill the same bowl? You try it now, Brian, just to make sure,' he said. Diane and Stanley (both 6) were playing with water and milk bottles of different sizes which had different coloured markings on them. They discovered that the one with the blue label was the same size as the ⅓-pint school milk bottle and Diane asked if she could write this on the blue label. This led Stanley to ask for the names of all the bottles and he wrote them on the labels. The names seem to hold very little significance for them but one day Diane was heard to say, 'The pint bottle fills the half pint bottle twice, so the big bottle must hold twice as much as the little one.'

As in the case of linear measure there may well be some children who are ready for more advanced weighing and capacity measurement before they leave the Infant School. The use of recipes provides one answer to this and certainly many children thoroughly enjoy cooking operations, particularly if it is possible to have the results put into the school oven and taken home at the end of the day.

The report of the Mathematical Association[1] contains an interesting discussion on measurement, in which it is pointed out that the choice and use of units for measurement is not the obvious and simple matter it may appear to the adult mind. It is 'a comparatively advanced abstraction attainable only after ample experience of the various aspects of extension in a qualitative sense, followed by experiments with natural, varying standards such as stride, handbreadth'. This is the view accepted here with the additional suggestion that if children are given our standard measures to play with this experience may contribute to the act of abstraction when the children are ready to make it.

Relevant to this, of course, is the observation already made that the matching methods used for the measurement of different kinds of magnitude vary considerably, some being within the child's understanding earlier than others. Capacity and length are both measured by a visual comparison with a standard unit, though capacity comparisons are less direct and more difficult to see unless transparent containers made of polythene or glass are used. Area is also measured by a visual comparison and if it is first undertaken by finding how many squares fit into spaces it seems likely that the abstractions in the table of square measures will be understood more readily than they often are by Junior School children.

The concept of weight is probably more real to the child than any of the others because of knowledge he gains through pushing, pulling and lifting his toys and household furniture about. Nevertheless the understanding of what we are doing when we measure weight is more difficult to achieve, since the matching activity is more complex.

Fractions arise naturally in children's weighing and measuring activities and this seems as good a context as any in this book to consider what kind of concept of fraction children of this age can form. Integral or whole numbers served all the needs of early man, but occasions arise when there is a need to break the whole into parts. Our word 'fraction' derives from the Latin word 'frangere', to break. This word was adopted in England in about the thirteenth century when it supplanted the use of 'broken number' or 'part'. The Egyptians used a system of Unit Fractions in which the numerator is always, except for the special fraction $\frac{2}{3}$, expressed as unity. Thus if they wished to record three-quarters it would be given as $\frac{1}{4} + \frac{1}{4} + \frac{1}{4}$ or perhaps as $\frac{1}{2} + \frac{1}{4}$.

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[1] Mathematical Association, *The Teaching of Mathematics in Primary Schools.*

When dealing with weights and measures it was soon found necessary to have submultiples of the standard unit and a system of Commercial Fractions was created with smaller weights and measures. Examples of these are: the ounce, $\frac{1}{16}$ of a pound; the inch, $\frac{1}{12}$ of a foot; the minute, $\frac{1}{60}$ of an hour. In addition to these the Greeks invented a system of fractions for everyday use, called in Latin 'fractiones vulgares', from which we get our term vulgar fraction.

Halves and quarters enter into little children's experience as names for particular things almost as early as any other number words. A 'half' is what you get when you share a piece of chocolate with a brother or sister and to get 'the big half' is sometimes very important. A 'quarter' may be the little packet of tea wrapped up in silvery paper, and a 'half' the block of butter bought by Mummy at the grocer's. A 'third' of a pint of milk is the little bottleful which we drink every day at school. We go home from school at 'half' past three and we have coins which are called 'half-pennies'.

At first, as with other number words, these will be names for particular events, and will be used without any appreciation of their mathematical meaning. The development of the fractional idea will come from experience of material things being broken in pieces. Sharing a possession is usually accomplished by repeated halving which gives rise to a natural if limited idea of fraction. The discovery during play that the school milk bottle must be filled and emptied three times into the pint jug in order to fill the latter, and that the pint can be re-distributed into three such bottles, may help a child to realize the relationship of the part to the whole which gives meaning to the term $\frac{1}{3}$ of a pint. Similarly he may discover that two half-pound weights balance a pound weight, and that something which can be measured with half his ruler may be said to be six inches long or $\frac{1}{2}$ foot long. In this way many of the basic facts and processes in vulgar fractions can be discovered experimentally by children and this is where learning should begin.

The following examples illustrate the kind of understanding which can be expected where the environment is helpful. Janet and Andrew, both five years old, were looking at a picture which included a ladder broken into three pieces. Janet said: 'It's broken in half', but Andrew corrected her saying, 'No, it isn't. Half is two pieces and that ladder is three.' Christopher, aged six, asked his teacher how much milk there was in a school milk bottle and was told $\frac{1}{3}$ of a pint. Later he was seen counting the bottles in the rack—47 in all—and came to say that there were 15 pints and 2 left over. Anthony, at six-and-a-half years was measuring a desk with a footrule and found it to be $2\frac{3}{4}$ feet long. The previous day he had been using weights labelled $\frac{1}{4}$ lb., $\frac{1}{2}$ lb. and 1 lb. and had learned how to write a quarter, so he now wrote down the result of his measuring operation as $2\frac{1}{4}\ \frac{1}{4}\ \frac{1}{4}$ feet.

Anthony's method of dealing with fractions is one we have observed on other occasions and suggests that unit fractions are easier to understand and sooner dealt with than fractions which have a larger numerator and therefore express a more complex idea. To use the term $\frac{3}{4}$ the group idea of three and the part idea of quarter have to be held together. This does not happen till the child has come to understand both ideas through experiences in which one or other occurs by itself.

Discussing the teaching of fractions Miss Riess says:

> The child's stream of thought should flow in continuity from concrete wholes and their natural parts to abstract ideas of ratio and arithmetical fractions, but these ideas become muddled at their very

source by visual aids, blackboard diagrams, etc. Measurement still remains the critical activity for releasing in the mind of the child the formative notions pressing toward a mathematical concept of fractions.'[1]

I am in entire agreement with this statement and believe that the principle implied here can be illustrated in every field of number learning.

This chapter cannot be closed without some reference to the less tangible extensions which man finds it necessary to measure. The duration of time is perhaps the most difficult to understand and clearly beyond the comprehension of children of the age with which we are concerned. Early man devised water-clocks and sand-glasses, we have clock-time. Little children learn to read the clock face about the same time as they learn to read books but this does not mean that they understand the principle of clock-time. It enables them to be able to answer the question, 'What time is it?' and to recognize the patterns on the clock face which spell bedtime, Children's Hour, home-time and dinner-time. They may also learn to read calendar-time and discover that we have seven days in a week. Perhaps the situation easiest to comprehend is egg-time if Mother uses the kind of timer containing sand. Little children like to watch the sand falling from the higher into the lower container and will soon learn that when all the sand has fallen the eggs are ready. What he cannot be expected to understand are the various time-scales which man has invented, nor the principles involved in indicating positions on these time-scales. These are things which are likely to be of great interest to most children when they are older and are best left till then.

The instruments used for the measurement of speed, temperature, and atmospheric pressure are met with in the experience of the majority of young children today. Moreover they have experience of these dimensions and have some understanding in terms of their experience fairly early in life, as is shown by their behaviour.

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[1] Riess, A., 'A New Rationale for the Teaching of Fractions.' *Harvard Educ. Review*, Spring 1955.

Probably few children escape the experience of being told to 'hurry up' and soon learn to understand what mother means when she says this. Fast and slow become differentiated in running, walking and dawdling, and in the movements of trains and motor cars, and many children before they reach school age use these terms with understanding, and cause their toy vehicles to run at different speeds quite deliberately. Recently, when a tape-recorder was taken into a nursery class the children crowded round to watch. Their comments included a number of this kind: 'It's going fast,' 'That one goes fast and that goes slow,' 'It goes round and round,' 'Can you make it go back?'. There comes a time in the life of many children when part of the fun of going out in the car with Daddy is the opportunity it offers for watching the speedometer and the speed-limit signals, and seizing on any opportunity to tell the driver that he is doing 'more than 30', though just exactly what is meant by 30 miles per hour will not be understood till later.

Hot and cold are probably distinguished first as variables in things you are given to eat and children soon learn how to protest if the temperature does not suit them. The bath-water, too, must be just right, neither too hot nor too cold. Children know what it means to feel too hot and to shed clothes in order to get cooler, similarly they soon learn how to make their needs understood if they feel cold. There are thermometers for measuring body-temperature, room-temperature and water-temperature and most children will see these in use at home and at school. Having your own temperature taken when you are ill seems to fascinate or repel children, and often this incident is imitated later in doll play. The little child may watch mother using a thermometer when cooking, or for testing the bath-water for the baby, and is likely at some point to ask what it is for. If there is a thermometer in the classroom and teacher keeps a daily record of room-temperature in graph form on the wall beside it there will come a time when some children will take an interest in this ritual, will want to read the thermometer for the teacher and perhaps be prepared to be interested in the picture which shows how the room temperature goes up during the day. All this can be enjoyed and form a basis for the development later of the concept of room-temperature and its measurement in terms of the expansion of a liquid.

Some days are hot and some are cold; so also some days are sunny, some are cloudy; some are showery and some are very wet. These variables, too, are noticed by children because they affect their lives. They may prevent them from going into the garden when they want to, or from going shopping with Mummy. In homes where there is a wall barometer which Daddy refers to from time to time when the family is planning an outing or he has a gardening project on hand, there comes a time when the little child begins to ask questions about it and may learn to read the words dry, showery, wet, and look for the position of the hand in relation to them; but it hardly needs saying that Daddy will be wasting his time if he tries to explain the principles involved in measuring atmospheric pressure to children of the age we are considering.

At the close of this chapter some suggestions are listed for materials and experiences which will be useful to the children in connection with the kind of growth discussed in this chapter. They will be useful to the teacher only if she considers them in the context of the educational principles implied here. What one needs to be aware of first of all are the opportunities which the affairs of every-day living at home and school offer for learning to appreciate dimensions of magnitude, and the various extensions in the world of physical and human events to which man has to accommodate himself. As we talk to the children about these when the occasions offer themselves we give them the language for defining and classifying these experiences when he is ready. Secondly, there are the opportunities for extending understanding gained in this way by the careful planning of the children's play environment which should include a wide range of experiences and provoke the kind of experimental play that will lead them towards new discoveries and further abstractions. This environment includes not only a variety of play materials but also a teacher who is interested in what is going on, is prepared to be observant of children's behaviour and to inquire into what it can tell her about their development, to be alert to the children who do not seem to be learning much and those who are ready for more challenging experiences. She has to try neither to retard development by withholding opportunity for learning by the brighter children nor to cause frustration and anxiety in the slower children by trying to insist on learning for which they are not ready. This means providing an environment which makes it possible for each child to operate at his own level, to find experiences which challenge interest and experiment and offer opportunities for learning. This is not easy and implies the presence in the room of an alert observant teacher who knows how to evaluate the behaviour she is observing in terms of what the child is or is not learning.

The following quotations were taken from a book compiled by a group of seven-year-olds who seemed to find great satisfaction in collecting statistics of various kinds, but the range is due, I think, to the fact that their teacher shared their interest and helped them to extend it.

My daddy is the tallest person in our family. I am the smallest person in our family. There are 4 people in our family my mummy and daddy Miss —— and me.

In our family there are 3 people. The 3 people they are my mummy daddy and me. My daddy is the tallest person in our family. I am the smallest one in our family. My mummy is the middle-sized person in our family. My daddy is the fattest person in our family.

The smallest thing I know is one small tuft of moss. The sea is the largest thing I know.

Michael is the tallest boy in our class. Margaret is the tallest girl in the class. Jacqueline is the smallest girl in our class. I am tall but I am not as tall as Margaret.

The lowest thing I know is a creeper at the bottom of a deep well. The sea is the largest thing I know. The smallest thing I know is a crumb.

There are 6 rooms in our house. We have 3 rooms dowstairs. There is 3 rooms upstairs. The rooms downstairs is the kitchen, the dining room and the sitting-room. The rooms upstairs is the 2 bedrooms.

The smallest thing I know is an ant.
The deepest thing I know is a piece of coal at the bottom of the pit.
The highest thing I know is the sky.
A giant is the largest thing I know.

I have one sister in our house. She is 10. She is older than me. She is 4 years older than me because I am 6. I have one brother in our house he is five years old he is one year younger.

The narrowest thing I know is a piece of cotton.

I am the youngest person in our family.

Miss C——'s window is wider than Miss B——'s window.
In our window there are 60 panes of glass and 12 panes in Miss B——'s.

The widest thing in our room is the cupboard. The narrowest thing is the green stool.

The number on our back door is 15.

A load of coal weighs 1 ton.

I set off from school at 12 o'clock and I got home at 4 minutes past it took me 4 minutes to get home.

I can buy a pint of beer
I can buy a pint of milk
I can buy a pint of vinegar
I can buy a gallon of petrol.

When my mother buys a bag of coal from the coalman she has got 1 hundredweight.

The number of the bus from Wakefield to Pontefract is 70.

On Saturday April 21st they put the clocks forward a hour.

Some boys made a cafe table. I painted it. It is a box with four legs. Each leg is 18 inches it is 72 inches all together for the four legs.

I am 3 feet 10 inches tall. I said I bet Miss C —— is 5 feet 7 inches tall. Michael said she is 6 feet tall. She is 5 feet 3 inches so I was 4 inches too tall Michael was 9 inches too tall.

We thought Miss W—— was 5 feet 11 inches. We thought Mr. H—— was 5 feet 8 inches. He says he is 5 feet 7 inches so we thought He was one inch too tall.

We filled a pint bottle with warter we poured the water into the big bowl it just covered the bottom and looked just like a bit of water then we poured it into a small dinner plate it just filled it it looked a bit deeper next we poured it into the basin it looked right deep it was 4 inches deep we poured it into the pan it was 3 inches deep then we poured it in to the pint measure it just filled it it was 5 inchess deep we poured the water into the scales it was 1 and a half inches deep.

SOME SUGGESTIONS FOR CLASSROOMS IN AN INFANT SCHOOL

A trough containing water or silver sand with polythene cups and jugs of various sizes, spoons of various sizes, medicine bottles, polythene thistle funnels, a piece of rubber tubing, milk bottles in usual sizes.

A good assortment of bricks—large and small cubes, long bricks, flat bricks, and a variety of other shapes—squares, oblongs, circles, ovals, triangles, octagonals, etc.

A woodwork bench—assorted sizes and shapes of wood soft enough for children to work with. Tools including hammers, pincers, handsaws, clamps, screwdrivers, nails and screws—all in assorted sizes.

Balls of different kinds—hard and soft, solid and hollow, large and small. Also a bat or two.

Materials for domestic play—dolls which can be undressed, a doll's cot, a kitchen stove, a dresser, a wash basin, wringer, clothes line and pegs; cups and saucers, pans, kettle, teapot, stools and table, a toy telephone, dressing up clothes. (Many of these can be made from packing cases and need not be elaborate or expensive pieces of equipment.)

A bookstand easily accessible to the children, some books specially designed for counting and comparing activities, though many picture books are used in this way by children who are interested.

For outdoor play—a climbing frame, slide, see-saw, balls, hoops, skipping ropes.

Sandpit with spades and buckets of different sizes and shapes, a rake, a trowel.

For painting and drawing—easels, brushes of various thicknesses, thick and thin pencils, paper of various colours and shapes (so often it is only the oblong shape of white which is available).

Constructional toys—fitting toys in wood and plastic, jig-saws, mosaic sets, hammer toys, meccano, etc.

For modelling—clay and dough, patty pans, cutters of varied shape, sticks, feathers, shells, a blunt knife, a fork.

Available somewhere in the room (either on a shelf, in a cupboard or hanging up in a corner) things to measure with—and sets of Work Cards of graded difficulty for those children who have enjoyed a period of experimental play with the various measuring instruments and are ready to tackle some problems with them. Some examples are given overleaf as illustrations of what is in mind, but teachers will want to work out their own sets based on their knowledge of their children and the opportunities they can offer them. If each set in any series is written on a different coloured card the children can readily find those suited to their own level and progress to more difficult tasks at their own rates. Teachers may find it helpful to make pockets for the cards on a length of material which can be hung alongside the wall beside the appropriate apparatus.

(a) *For Linear Measurement*

Sticks of various lengths, including foot-rulers, yard-sticks and tape-measures, a trundle wheel and a pair of calipers.

Sets of graded measuring cards.
A book of simple knitting patterns.
A book of simple instructions for woodwork models.

(*b*) *For Weighing*

A pair of stout scales, soundly based, with a fulcrum showing when the pans are evenly balanced.
Jars or tins containing materials of different kinds for balancing and weighing, e.g. dolly mixture, peas, chestnuts, stones, silver sand, flour, bags of silver sand or grit weighing 1 oz., 2 oz., etc.; ¼ lb. and ½ lb. packets of groceries such as tea and sugar.
Boxes of different sizes filled with a variety of substances, e.g. lead shot, sand, flour, cotton wool—made up so that the children may discover that weight is not correlated with size.
For more advanced children other forms of balance, e.g. scales for finding their own weights, a letter balance, the kind of spring balance used in many modern kitchens, a spring balance from which parcels can be hung.
Sets of graded weighing cards.
A book of simple cooking recipes.
A Post Office where parcels and letters are weighed and stamped.

(*c*) *For Capacity Measurement*

A zinc trough to hold water or silver sand.
Jugs, cups, milk bottles, a bucket, a gallon tin, medicine bottles, spoons of various sizes, thistle funnels.
Where possible it is helpful if these can be made of transparent material such as glass or polythene. Some should have the standard measures marked on them, others can be plain.
A book of cooking recipes.
Sets of graded measuring cards.

(*d*) *For reading other measures*

A clock at a level which makes reading easy.
A thermometer, and for older children a graph showing room temperatures at different times of day. If there is one indoors and one out-of-doors comparisons can be made.
A calendar.
A rain gauge.
A barometer.
A sundial.

A weather-vane.
A spirit level.

Examples for Graded Sets of Measuring Cards

Set I (*a*) How long is your table?
(*b*) How wide is the window sill?
(*c*) How tall is your friend?
(*d*) How high is your table?

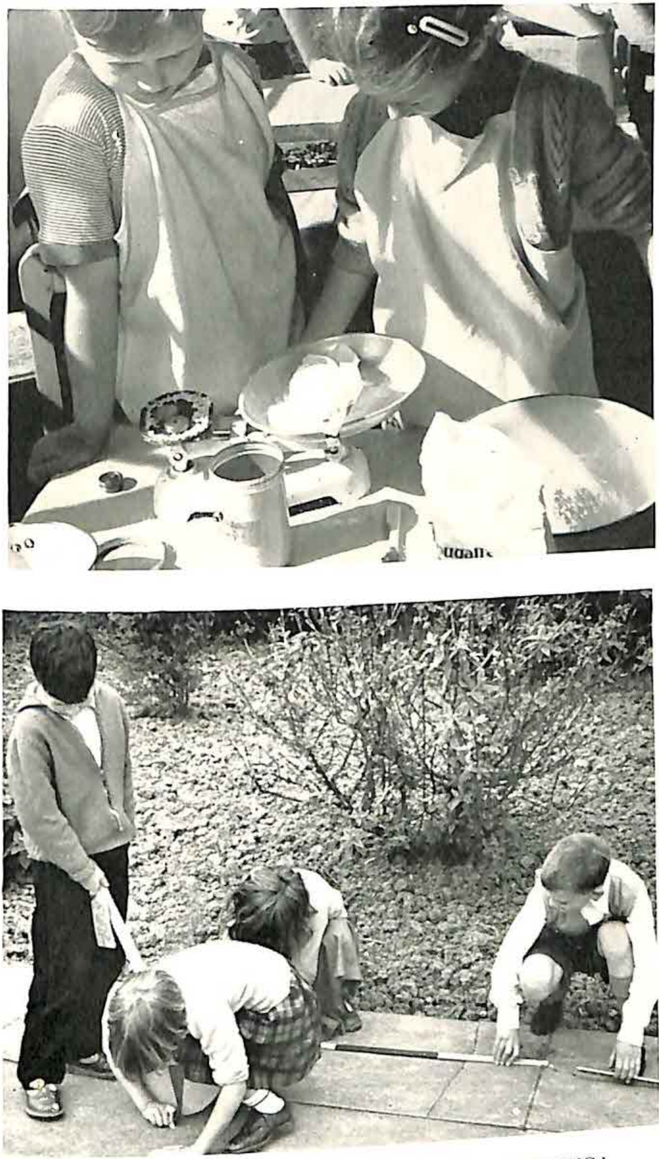
Set II (*a*) Measure the lines on this card. What is the colour of the longest line? What is the colour of the shortest line?
(*b*) Use a ruler to make a drawing of a ship to these measurements: length of hull 10 inches, depth of hull 2 inches, height of funnels 4 inches.
(*c*) Cut off 6 inches from the roll of red sticky-backed paper, two 4-inch pieces of blue sticky paper, one foot length of yellow paper, and as many strips of green paper as you like. Use these to make a pattern on a square of white paper.

Set III (*a*) Build a wall twelve bricks long. Make it three bricks high. Measure the wall and write down how high it is and how long it is.
(*b*) Draw a line a yard long on the floor. How many big bricks are needed to cover it? How many little bricks will cover it? How long are the bricks?

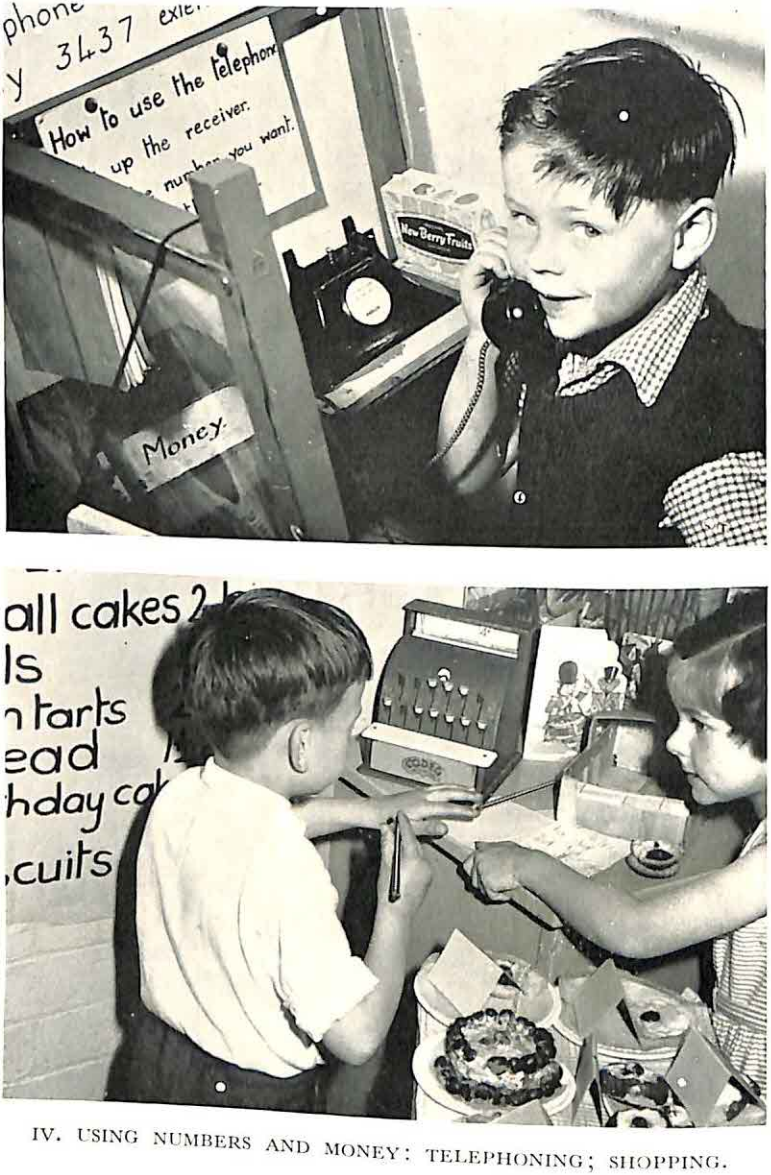
Set IV (*a*) Use the trundle wheel to find out the length of the corridor.
(*b*) Use a tape-measure to find out who has the smallest waist in the class.
(*c*) Use the wall measure to find out who is the tallest child in the class and who is the shortest. How much difference is there between them?

Set V (*a*) Guess how tall your friend is. Measure him and find out how near you were.
(*b*) Guess how wide the door is. Measure and find out how near you were.

Set VI (*a*) Use the calipers to measure three things in the room. Check your answer by measuring with a ruler or tape-measure.



III. USING UNITS OF MEASUREMENT: WEIGHING; MEASURING.



IV. USING NUMBERS AND MONEY: TELEPHONING; SHOPPING.

(*b*) Who is taller—your friend or you? By how much?
(*c*) How fat is the fattest book in the Book Corner?

Set VII (*a*) Use the Knitting Pattern Book to make a striped scarf for your doll.
(*b*) Make an aeroplane using the measurements given in the Models Book.

Examples for Graded Sets of Weighing Cards

Set I (*a*) How many peas balance 3 chestnuts?
(*b*) How many stones balance a cupful of silver sand?

Set II (*a*) How many toffees weigh 1 oz.?
(*b*) How many stones weigh 4 oz.?

Set III (*a*) Guess how many peas weigh 1 oz. Use the scales to find out how near you were.
(*b*) Guess how much a cupful of silver sand weighs. How near were you?

Set IV (*a*) Weigh the little white parcel and the large green one. Which is the heaviest? What is the difference between them?
(*b*) Use the little white 1 oz. bags to find out how many ounces together weigh 1 lb.?

Set V (*a*) Mrs. Smith has ordered ½ lb. of butter, 4 oz. of toffees, 2 lb. of apples, ¼ lb. of flour, and ½ lb. of tea. Put the goods ready in a bag for her and don't forget the bill.

Set VI (*a*) Use the Recipe Book to make some biscuits for tea today.
(*b*) Use the Recipe Book to make some fondants for the party. We shall want twice as many as usual.
(*c*) Use the Spring balance to find out which is the heaviest parcel in the box.

Examples for Graded Sets of Capacity Cards

Set I (*a*) How many cupfuls fill the green jug?
(*b*) How many spoonfuls fill the white cup?
(*c*) How many jugfuls fill the bucket?

Set II (*a*) How many tablespoonfuls can be taken from the large medicine bottle?
(*b*) How many pint-jugfuls fill the gallon tin?

Set III (*a*) How many times do you have to fill the $\frac{1}{2}$ pint milk bottle to get enough water to fill the pint bottle? Use the funnel so that you don't spill any water.
(*b*) How much water does the large jug hold?

Set IV (*a*) How many pints are there in a gallon?
(*b*) How many pints are there in a quart?
(*c*) How many $\frac{1}{3}$ pints are there in a pint?
(*d*) The doctor has ordered one teaspoonful three times a day from the small medicine bottle. How many days will the medicine last?

Set V (*a*) Use of the Recipe Book to make some scones for tea.

INTERESTING BOOKS FOR THE OLDER CHILDREN

Hogben, L., *Men Must Measure*, Rathbone Press.
Smith, T., *The Story of Measurement*, Blackwell.
Goaman, M., *Judy's Cookery Book*, Faber.
Judy's Sweet-making Book, Faber.
Judy's Knitting Book, Faber.
Hutchinson, M. M., *Making and Keeping a Bird Table*, E.S.A.
Making and Keeping a Box Garden, E.S.A.

CHAPTER EIGHT

FROM COUNTING TO CALCULATION

COUNTING is the process which enables us to answer the question 'How many?' as compared with the methods of measurement being considered in the previous chapter which provide answers to the question, 'How much?' In the latter case we are concerned with continuous quantity, situations of magnitude and extension; in the former we are dealing with discrete quantity, situations of multitude—with collections of items and with series.

We have already considered in an early chapter the nature of the matching and ordering operations which we call counting. We noticed that at first man matched with something quite concrete such as stones, marks on trees or clay. What he was doing here was simply making a new collection which matched the other one, but was easier to handle. Later when he learned to use his fingers and other parts of his body for matching he was matching members of a known group, e.g. the fingers of one hand, to members of the unknown group. This enabled him to say, 'As many arrows as I have fingers,' just as the cricket umpire today, dropping pennies from his hand into his pocket, is saying in effect, 'As many pennies in my hand as there are balls still to be bowled in this over.'

In counting we are using words for matching instead of things, and this marks a big advance in mental operation from the matching methods just referred to. A different sense modality is involved and the difficulty of arriving at a real understanding of what is involved in counting is not always appreciated by teachers. The difficulty with words is that you cannot see or feel them and, as we know, many children and some adults find it necessary to use fingers while saying the words, to make the situation more real.

Three things are necessary if we are to use words for counting. First we must have *a set of words* each different from the others. Secondly we must always *use them in the same order*, thirdly we must *match each to an item in the group* we are counting. Many children coming to school at five will know a number of words in the series and be able to speak them in their correct order; but only a very few understand how to use them as matching instruments. This can readily be checked by watching how children count sets of beads, bricks or biscuits. Some will match off with one, two and three, but very few much further than this, and many not as far.

This understanding will come in time if children have experience of watching adults counting, of counting with them, and of having a number of different kinds of things to count. Most children enjoy sorting things into groups of the same kind, making collections of beads of the same colour, shells of the same shape, etc. Most children enjoy putting things together which belong together—cups with saucers, spoons with forks, hats with coats. Here is another matching operation. Most children also enjoy arranging things in an order—bricks according to size, rings on a rod according to size, fitting bricks inside each other, etc. *Sorting*, *matching* and *ordering* are all involved in counting and these activities will help to bring him nearer the act of abstraction he has to make sooner or later.

If the pegs in his cloakroom are numbered he will soon learn to recognize the symbol attached to his peg, but probably earlier than this he will know whereabouts in relation to the other pegs his is situated. One day the reason for numbering his peg with a five will be understood, and likewise the reason why his home is called Number Seven, though this may be more difficult to grasp owing to our habit of numbering houses in odds and evens.

The fact that many children persist in using fingers for matching long after they are able to count with words suggests how difficult they find it to get away from the concrete level of operation, and it is debatable whether we should discourage it as some teachers do. While the child needs this to help his thinking we should surely allow it.

When he uses his fingers he has in fact taken *one step away from the concrete situation.* Other children will find it more helpful to use beads or counters for this first step. This simple act involves more than is sometimes recognized. It involves imagining that the beads, counters or fingers are representing items in the collection he wishes to count.

Another idea which the child has to attain to is the *concept of the group*. Making collections and counting them, playing with games like skittles and dice games, laying tables for meals, experiences of the family group will all help to contribute to the attainment of this concept, but it is not present at first as the following example illustrates. Margaret, aged five years, has pocket money given to her on Saturdays. On one occasion Auntie was present when Daddy gave her the usual three pennies and she contributed three more. When asked how many pennies she now had, Margaret counted the whole collection in her hand before saying 'Six'. Now, in a similar situation the older child or adult would count on from the known group of three, three more places to reach six, but although Margaret will tell you if you ask her that Daddy gives her three pennies on Saturdays she still does not think of them as a group but as three distinct items. We could give numerous examples to illustrate this point, e.g. when assessing the number of skittles knocked down, if a certain number are knocked down the little child may be able to record his score by making a tally in the form of marks on a board, or by matching the number with shells into a jam jar, but if another one falls before the end of his turn he will not simply add one more to his tally but start again from the beginning, counting the skittles and then matching with marks or shells. It is in fact quite a long time after counting has begun before the child is *able to count on* when faced with a situation which could be written as $5+4$.

Some counting situations are much less concrete than Margaret's. We count on each stair as we go upstairs, or we count the coat pegs along to our own, but the whole situation is not so clearly defined in the child's mind that he can recognize that when he reaches No. 6 he has behind him a collection of five.

I have known teachers who meet this difficulty by *telling* children how to deal with these situations, explaining that it is unnecessary to count the five, and then doing little sums with them until a verbal habit of counting-on is established. This is not difficult to do, but in my view it is not the most helpful way of helping forward little children's learning. Until through many concrete experiences the idea of cardinal five is established as a concept he will not understand *why* he is being required to work out little sums of this kind one way rather than the other, whereas when he has reached the stage of using these numbers as group values he is likely to begin counting on from the five, or whatever number is involved, quite naturally. The teacher may need to point out to some children *at this stage* that there is no longer any need to begin counting from one, but if they do not take the hint this would be an indication that they are not ready.

There are other situations in which counting is fairly common which present another kind of difficulty. If you set out to count the number of steps you have to take to get from one side of the room to the other, you are counting movements, each of which slips away into the past as soon as it is taken. The same thing happens if you count the number of times you can skip without stopping, or bounce a ball without losing it.

Children need to experience all these different kinds of situations in which enumeration can be meaningful, and with the teacher's help will learn from them. Most children enjoy number rhymes and jingles but they are less likely to be simply a repetition of words if the concrete situation is present at the same time. If children represent the Ten Little Nigger Boys and one disappears at the end of each verse then both actors and observers will have an opportunity to learn more about what 'one less than ten' etc., means. Acting out the story of the Three Bears gives an experience of threeness and of size-order. The story of Noah's Ark can be used in a similar way to give an opportunity for counting in twos. We need to consider carefully what basic number notions are enshrined in these well-loved stories and rhymes and make full use of the opportunity they offer for adding to understanding. A board covered with some kind of adhesive material like flannel on to which pictures can be built up by putting on and taking off items at appropriate moments is another way of making the situation more concrete, and little children enjoying building up and altering pictures in this way.

In addition to this, scoring games like skittles, fishing, getting balls into a basket and playing home games with dice can all help to enrich experience of groups and series, paving the way for the abstractions necessary before arithmetical calculation can be meaningful, but, as Miss Adams[1] has wisely pointed out, the value of scoring games as a numerical experience will depend first on whether the children are genuinely interested in the game (and not all children are!) and secondly on how important the score is felt to be. Some of these games can be played by yourself when the interest would be in beating your own record, or if played with someone else then it is beating theirs. Little children will require some concrete representation of scores in order to make these comparisons. There are various tallies which can be used—marking each point won with a chalk mark on a blackboard, by putting a pea or milk straw in a jar, or by putting a bead on a rod. The advantage of the last suggestion is that it makes comparison easier as Miss Adams shows. She makes the point that comparisons are generally of greater interest than totals, and with this I would agree.

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[1] Adams, L. D., *A Background to Primary School Mathematics.*

A further point which needs to be made about scoring games is brought out by a story told against herself by a student in training. She had made a beautiful fishing game for school practice. Half the fish were cut out of pink card and half out of blue, and each was carefully numbered. She explained to the children that to play the game you fished up one blue and one pink fish and then added the numbers to find what your score was. Later she returned to the table where two children were playing with the game. Brian, aged five, had caught a 3 and a 5. However, when the student asked him, 'How many have you caught, Brian?' his immediate reply was 'Two', pointing to the fish in front of him.

About three years ago I visited a young teacher and her class of five-year-olds and found forty-five of them all busily making up sums with spinning tops. You had two turns with your top which gave you two figures which you then wrote down and added together.

Both these young people had failed to distinguish between games which enable children to acquire an understanding of number relationships and practice games which can be useful in establishing these relationships as part of the memory store. In actual practice, I think many experienced teachers would agree that when children have acquired the necessary concepts they do not require the motivation of a game. They love doing pages of sums and this activity provides all the practise they need. In cases where this is not so, games like lotto, tunnel ball, quoits and darts are useful.

The stories told on pages 7 and 8 may suggest games and activities which contribute to the development of number concepts. As well as scoring games many activities such as bead-threading, building towers, and making collections of things evoke comparison and draw attention to number values, as in the case of four-year-old David who was threading beads and said, 'Look, my thread is longer than Jean's, 'cos I've got eight on mine and she's only got three,' or another four-year-old who counted the bricks in his very tall tower to sixteen and was then heard to say, 'My! that's a lot,' or five-year-old Christine who was playing skittles with Philip and said to him, 'I've knocked one down. That means there's five you can knock down.'

Counting in groups is a useful experience and there are many daily-life experiences of things occurring in groups which can be used for this, e.g. for counting in twos—children's legs, arms, etc.; for counting in threes—school milk bottles to a pint, 3d. pieces to shillings; for counting in fours—chair and table legs, car wheels; for counting in fives—fingers, grouping of minutes on the clock, arrangement of bottles in the milk crate; for counting in twelves—pennies to shillings, inches to feet. One little boy who had been invited by his teacher to say how many wheels there were in a group of cars shown in a picture took great pleasure in the fact that he arrived at a different answer from the rest of the group. 'I counted in the spares,' he explained.

What developments are we looking for through experiences of this kind? Familiarity with the number series, development of ability to make a one-one correspondence between objects and number words, ability to think in groups as well as in units, understanding how to answer the question 'how many?' in respect of a group, understanding of how to give positional value to an item in a series, ability to add one small group to another small group and arrive at the size of the new group, ability to take away items from a small group and find how many are left, ability to compare one group with another, ability to count in groups so many times, ability to share a group between a number of individuals, a discovery of some of the relationships and patterns in the number series and an interest in these.

What is especially important for all later work is the development of a mode of relational thinking with regard to numbers. Seven stands in a particular relation to all the other numbers in the series: to six and eight, to seventy and to forty-nine, to three-and-a-half and to one-and-three-quarters. It can stand in a variety of relations with other numbers to give a variety of values, e.g. 70, 7, 701, $\frac{1}{7}$, 7^2, 2^7, $7+4$, $7-5$, 7×2, $7+3$. These relations exist by virtue of the structure of the number system which man has created, and they illustrate the power of the number series. This is the tool, the basic grammar of the language which children will be using in all later arithmetical operations. If they have come to understand it as a pattern of relationships and have developed an appreciation of its simple beauty and order they will be saved from many of the errors of thinking which lead to the kind of results exemplified in chapter 1 by the division sum of a nine year old boy or the College student's attempts to resolve a simple fraction sum. A correct initiation into the language of mathematics would make mistakes of this order impossible because they would 'feel' wrong. This initiation begins at home and in the Infant School; for many children it is not completed by the time they reach the Junior School, as teachers will find if they care to try out some of the items in the Number Readiness Test with children between the ages of eight and eleven years.

In passing we might notice that this mode of thinking can be established without knowledge of the written symbols, and for most children will be the result of concrete experiences of the kind we have been talking about, experiences shared with the teacher and other children. The operations are manual operations with real things accompanied by thought and language, but very little will be possible at the mental level without the accompanying concrete experience. We should not be in any hurry to get children through this stage to more abstract number operations. The understanding which they gain through experiences of this kind, and probably only through such experiences in most cases, is invaluable to all later understanding, and we must not deprive them of the opportunities for development which they afford. Some children may need most of their Infant School life for this level of learning, and in these cases we must see that the experiences are sufficiently varied to hold their interest, while remaining at the same simple concrete level as long as there are children who need it.

Suggestions at the end of this chapter may be helpful. Most of them can be varied in many ways by the imaginative teacher. Children of seven years old should not be given exactly the same kind of experiences as they had when they were five and six for obvious reasons, but there are in many classes of seven-year-olds, some children who are still unable to operate at a higher level than is characteristic of most five-year-olds. It is vitally necessary to maintain the interest of these children and variety and new activities are essential. This principle is not always recognized by teachers responsible for planning worthwhile classroom and playground activities.

It is probable that most children will need help at some stage to systematize the knowledge about numbers which they have been acquiring through the variety of experiences we have been discussing. What is important before they are introduced to the first easy steps in Arithmetic is that they shall have begun to appreciate the structure of our number system; its relational character and the principle inherent in the system. Children who have developed the habit of thinking relationally about number facts will have little difficulty with arithmetic since all we are doing in our calculations is to unearth some of the less readily accessible relations inherent in the system.

There is unlikely to be general agreement among my readers that this is a need to which teachers must attend. Many Infant teachers seem to feel that their job is to provide a wealth of experiences which will provoke children to count and measure, to compare, to group, to share, etc., and that the concepts will in time be formed in the natural course of things. The following statement from the Ministry's Handbook of *Suggestions for Teachers* seems to lend support to this idea: 'Where a child has freedom to experiment in a suitable environment the development of number ideas occurs spontaneously without formal teaching and without difficulty.' The numerous examples of children's learning recorded in this book provide evidence in support of this view, but experience suggests that the ability on the part of the teacher to evaluate the suitability of the environment, and the correct timing of *informal* teaching, is vital for the generation of number concepts in all but the really able children.

We have already referred in chapter 6 to the various forms of structural material which are now on the market. All in their different ways are designed to foster an appreciation of numerical relationships and establish the habit of relational thinking in respect to numbers, and the suggestion of the present writer is that some of this structural material might well find a place in our Infant classrooms. It could be used as experimental play material for the young children and because of its design might well foster relational thinking, but it might also be used with the teacher to make explicit and systematize all the varied ideas about number and quantity which the children have been developing as a result of their very varied daily life at home and at school. The age at which this would be appropriate will, of course, vary from child to child.

Let us now suppose that we have a group of children who by the way they handle quantitative situations of various kinds suggest that they are now ready to undertake operations with numbers at the abstract level in what is usually called Arithmetic. Let us suppose we have watched the way they use the number series for counting and numbering, and have satisfied ourselves that a relational type of thinking about numbers has been established. How should we help them into the first simple operations with numbers so that they are undertaken with understanding and not as applications of a rule of thumb?

Before attempting an answer to this question, however, we might consider the value of applying tests of number readiness in cases where we are not sure. Teachers who have become practised observers of children's behaviour, and skilful in assessing what it can tell them about levels of understanding, will not need to use tests of this kind, but in these days when classes are so large many teachers find it difficult to get the information they need and may find a test of readiness helpful. It may also be useful to those readers who are only just beginning to develop in themselves an understanding of the nature of number which we wish to develop in children. Suggestions for a test of this kind are given at the end of this chapter. It is not intended as a formal group test and items from it could easily be applied to individuals or groups of children in an informal setting.

Whether such a test is applied or not, what matters is agreement with the principle that children should not be asked to undertake abstract number operations until they are ready. After all they are going to spend anything up to an hour a day doing Arithmetic in the Junior School, and it seems much wiser to think of the Infant School period as one which gives you an opportunity to learn to understand what you can do with numbers at a level of experience in which you can operate intelligently.

Let us now return to our original question and ask a further one. What relationships between numbers, and what operations with them, are involved in the four basic processes of calculation in Arithmetic? We shall start from the proposition that all these are counting operations. In *addition* we are counting on beyond a known point in the number series; in *subtraction* we are sometimes counting between two known points and sometimes counting back from a known point. In *multiplication* we are counting on in groups instead of in ones, and in *division* we are counting back in groups. All these operations can be carried out at a purely abstract level in which we are operating with pure numbers with no concrete situation of any kind in mind. On the other hand many of the operations with concrete materials described earlier in this chapter involve one or other of these processes. If John has three marbles and Dick gives him two more, he has added to his collection and now has five. Dick, on the other hand, who originally had nine marbles now only has seven, since he has subtracted two from his group. Every time the children count in groups they are in fact multiplying, and every time they share a collection among themselves they are dividing.

What we are now concerned with is the big step, or series of steps, from the latter concrete operation to the former purely abstract operation, and most children will need some kind of counting-aid during the period when they are making this transition. What they need is something which will help them to take the next step away from the concrete situation. Many such can be thought of: a box of counters, a string of beads, a counting frame carrying movable beads, structural material of the kind devised by Catherine Stern,[1] Cuisenaire's coloured blocks,[2] Dienes' material, a ladder, and of course the counter's own fingers. It seems wise to have a variety of these readily available for children's use, as some are likely to find one kind more helpful than another as an aid to their own ways of thinking about and ordering their experience.

In the writer's view the ladder has certain characteristics which make it a particularly valuable counting aid at this stage, when one of the desirable developments is that the children shall become increasingly familiar with the number series which they will be working with in Arithmetic. Dr. Stern had the same need in mind in devising the Number Track which forms part of her structural material and since she discusses its uses very fully in her book, I shall confine myself here to an analysis of the development of calculating skills with the help of operations on a ladder. For some of these ideas I am indebted to Mr. Schiller, though I should not wish him to be held responsible for the use I have been able to make of them.

Before proceeding to examine some of the simpler Arithmetical processes in some detail let us be quite sure of the reasons why most children will need a concrete aid at this stage. Piaget's work has shown quite clearly that children do not reach the stage of mental ability which is required for operating with abstract numbers as such till much later than has usually been supposed, and what is being suggested here is that they should be offered an experience which helps them to take one step away from the purely concrete—a model of the number series on which to operate. Every time a child uses a counting aid to solve a problem he has to transfer the values in the concrete situation to the aid, whatever this may be. This requires a considerable act of imagination. The beads on the string, the counters, the fingers, or the rungs on the ladder have to be imagined by him as standing for the things to be counted. One day he will be able to use the number words in this way without a concrete aid, but for many children this will not come easily or quickly. In my view, children should be allowed to use aids as long as they want to. It will be found that they are discarded as soon as they are no longer useful to them. A ladder has one obvious advantage over the others mentioned in that it is a visual representation of the number series. Every rung represents a numerical position, and it is possible to move up and down the ladder in various ways. By experiencing the number series in this form, it seems likely that the children will be helped towards an appreciation of the abstract series of words which they will be using in all kinds of mathematical operations at a later stage.

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[1] Stern, C., *Children discover Arithmetic.*

[2] Cuisenaire, G., and Gattegno, C. *Numbers in Colour.*

We shall now consider the four basic processes in more detail. Addition enables us to deal with two rather different-looking situations. E.g. (i) I am in a street where the houses are numbered in the order one, two, three, etc. and I am standing outside No. 3. If I go on four more I shall arrive at No. 7. E.g. (ii) I have got 3 pennies. If Daddy gives me 4 more I shall have 7. In both cases the child of whom we are now thinking is able, like Caroline, to arrive at the answer without the concrete situation being present. But, as we have already noticed, the younger child is unable to count on in this way and experience with the ladder may help him towards this. Faced with either problem given above the child will put his finger on the third rung from the bottom and ask which rung will I be on if I go up four? This will tell him that 3 and 4 more makes 7, or if I start at 3 and count on 4 I arrive at 7. If he writes this down as $3 + 4 = 7$, the signs describe what he has done on the ladder, though there seems very little value in writing down calculations of this order.

The subtraction process is used to discover the answer to four very different-looking questions, which can best be examined by referring to concrete situations in which they arise.

(i). We need 8 paint brushes but only 4 have been put out. *How many more* do we need? This question can be answered by complementary addition, i.e. *building up from* 4 *to* 8.

(ii). Mary had 8 chocolates. She gave 4 away. How many are left? Here it is the *remainder* which is required and we usually think of this as a situation in which we *take away* 4 *from* 8.

(iii). I can get two single tickets for 4d. each or a return ticket for 6d. How much do I save if I buy a return ticket? Here the situation is one of *difference* and the question is which is more 6 or 8? We should probably do this by *counting between the two known points* to discover the difference.

(iv). John has 12 sweets and I've only got 5. How many more has he got? Again we should count from 5 to 12 or back from 12 to 5 to make the *comparison.*

If now we think of these situations in terms of operations on the ladder and what they would look like if we wrote down what we had done, they might appear something like this:

(i). I started at rung 4, I counted on 4 to rung 8. $4+4 = 8$.

(ii). I started at rung 8, I counted back 4 to rung 4. $8-4 = 4$.

(iii). I started at rung 8 and rung 6, and counted between. $8-6 = 2$.

(iv). I started at 12 and counted back 7 rungs to rung 5. $12-7 = 5$.

Notice that the answer occurs in different places in the equation according to the way in which the situation is being looked at and handled. Different situations provoke different questions and therefore different ways of looking at the problem to be solved. To call all these 'take away' sums is just nonsense. Children should meet all these situations many times with small quantities where the calculation is fairly simple before they are required to deal with similar problems involving larger number values. In the approach advocated in this book the child will already have met situations similar to these, and used the operations as a necessary means of carrying out something he wants to do or to find something out he wants to know. Now with the help of the ladder and the accompanying verbalization of the operation, the relations being manipulated will become more explicit, and any number fact discovered is seen as part of the pattern of the whole structure. There are eighty-one basic addition facts and associated with each are two subtraction facts with the exception of the doubles, eg. $3 + 3 = 6$, making two hundred and thirty-two facts in all. But the child who is being taught to think relationally will come to appreciate that only forty-five need to be committed to memory. If an addition fact, e.g. $4 + 2 = 6$ is known, so also can be the facts $2 + 4 = 6$, $6 - 4 = 2$, $6 - 2 = 4$, though many infants will not be able to appreciate this.

Put in another way one can say that the 2, 4, 6 relation can be expressed variously as:

$$\begin{array}{ll} 2 + 4 = 6 & 4 + 2 = 6 \\ 6 = 2 + 4 & 6 = 4 + 2 \\ 6 - 2 = 4 & 6 - 4 = 2 \\ 2 + 4 = 4 + 2 & 4 + 2 = 2 + 4. \end{array}$$

This principle is inherent in the structure of the ladder and in Stern's Number Track, and will be understood in time through operations on it. What may be necessary at some stage is to bring these known facts out into the open so that the child recognizes the facts he must commit to memory in order to facilitate his adding and subtracting operations. The chart in Fig. 13 is an explicit statement of these facts and it may be helpful to present him with them in this form at some stage.

The process of multiplication is also called for in more than one situation. Suppose, for instance, that there are six cars and we wish to ascertain the number of wheels. We might, of course, go round each car in turn counting each wheel till we arrived at the total number, but we could establish the number of wheels on one car which would give us the size of group which must now be counted six times. This operation could be done as an extended count of $4 + 4 + 4 + 4 + 4 + 4$ or by counting in groups of 4 six times: 4, 8, 12, 16, 20, 24. This illustrates the principle underlying multiplication, namely that it is a way of dealing with groups instead of units, and children who have been used to counting in groups in ways similar to those mentioned earlier will not find this kind of multiplication at all difficult to understand. They are becoming familiar with the multiplication sequences and many children will enjoy the patterns made by counting in twos, threes, fives, tens, etc., while at the same time they are learning the table of twos, etc. Furthermore as they use money and measures in shopping and measuring activities some of the individual facts in these tables occur so frequently that they become part of the memory store. The experience of giving and receiving change for coins of different values helps children to learn that four threes make twelve, two sixes make twelve and five sixes make thirty. Similarly through using rulers and tape measures they come to know that three twelves make thirty-six, through measuring liquids that two twos make four, and through reading the clock face that twelve fives make sixty. A number of such facts will be known before the child is ready to understand the equation form in which they can be written down, and which facts become known will of course depend upon the experience and interest of any particular child. It is the teacher's responsibility to see that this knowledge of multiplication facts is extended before he is required to undertake multiplication operations with pure numbers. Also she must decide when he is ready to learn to use the written statement with understanding. Many infants are not up to this.

So far we have considered multiplication only as an economical substitute for an extended count, i.e. by counting in groups instead of in ones. But multiplication is also used to effect an increase. We may wish to make a cake twice as big as the one we made last week, or to make a mixture which will produce three times as many fondants as we had last time. This involves multiplying each measure by two or three, and we often refer to these operations as doubling or trebling respectively. This kind of multiplication is also used to compare the size of one quantity with another. We may refer to one parcel as being three times as heavy as another, or one jug as holding twice as much liquid as another, and in each case can use this comparison to determine the size of one measure if the other is known. Children who are introduced to situations requiring comparison, and who are encouraged to notice situations of increase, will be prepared for this kind of multiplication through discovering meanings for such phrases as 'twice as many', 'four times as much', 'half as big'. This kind of experience provides a basis for learning for example the two times table in contrast to the table of twos discussed earlier.

Teachers do not always appreciate the necessity of planning experience so that children acquire both kinds of multiplication table. Most children learn the table of twos, etc., while they are in the Infant School and many are handicapped when they meet multiplication sums such as 62 × 4 for the first time since what is needed here is familiarity with the facts given in the four times table. The writers of the Mathematical Association's report draw attention to this point in the following statement:

> While children derive satisfaction from knowing a whole table, and knowing that they know it, too much emphasis on the rhythm sometimes means that the facts are not readily available when wanted one at a time, so that care should be taken to see that they are used in simple practical situations of the 'problem' type; if the numbers are

kept small such a situation brings out the purposes of multiplication and division devices while not dissociating calculation from experience. *Care must be taken, however, to ensure that right answers are being obtained by the new process and not by the less economical methods of addition and counting.*[1] (The italics are mine.)

This means, of course, thinking out problems involving situations of increase and a careful analysis of the method a child has used in arriving at his solution as well as an assessment of his answer as right or wrong. Where this is done teachers will find that some children need much more help in establishing the habit of using the more economical method than is sometimes appreciated.

Traditionally the approach to multiplication has been to memorize the tables first and then to proceed to use these in multiplication sums. Here a rather different order of approach is being suggested. When children are counting in groups, finding the sizes of groups, assessing total quantities containing numbers of groups of varying size; and when they are doubling, trebling or effecting increases with higher number values they are in fact undertaking multiplicative operations, and at the same time acquiring through their constant re-occurrence certain number relations as known facts. The teacher who is alert to the need for this kind of experience will find many situations occurring naturally in the course of everyday life and play activities which lend themselves to ordering operations of this kind, and can arrange for a gradual increase in the number of multiplication facts committed to memory. An important development comes with the recognition of equivalent facts in the tables, and a game with pennies might be helpful for this. Suppose we arrange a group in the following pattern:

o o o o
o o o o

and now ask how many we have altogether. Our answer should be eight but we can arrive at this by different routes according to the way in which we split up the group. If we look at it as a number of groups of two then we multiply by 4, i.e. 2 × 4 = 8, the symbol × being translated as 'multiplied by', but if we split up the pattern into groups of four then we multiply by 2, i.e. 4 × 2 = 8. In each case the answer is 8, thus showing the equivalence of 4 × 2 and 2 × 4. With games such as this an appreciation of equivalent multiplication facts should develop easily, and once the principle is understood it is a short step to the recognition that only half the facts contained in the tables up to the 12 times table need to be committed to memory. This is something many Juniors do not appreciate.

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[1] Mathematical Association, *The Teaching of Mathematics in Primary Schools.*

With this kind of basic experience children should find the transition to the more abstract operation of multiplying with pure numbers much easier than some do at present. With the help of such concrete aids as the ladder or a number track, and a chart which systematizes what they need to know (see Fig. 14) further learning is simply a matter of extending knowledge of number relationships and undertaking more difficult operations involving more than one step.

If situations calling for division are examined the same kind of variation is found. One may ask (i.) How many teams of 6 can we have if there are 48 children in the class? (ii.) If we want to share 48 sweets fairly between 8 children, how many will each get? (iii.) How long will each hair ribbon be if a piece of ribbon 48 inches long has to be shared between 4 children? Faced with the situations in real life they would be solved in different ways. No. i. would probably be solved by making up teams of six until all the children were placed and then seeing how many teams this gives. This would be a series of subtractions of groups from the total. Groups of 6 have been taken away from 48 eight times.

Problem ii. would most probably be dealt with by dealing out the sweets in ones to the group of 8 children till all were exhausted. Here groups of 8 are being built up till 48 is reached. In terms of a ladder operation this might read: 8 × 6 = 48 or 48 ÷ 8 = 6. To solve problem iii. we should, no doubt, fold the ribbon into quarters, cut at the folds and then measure one piece. In terms of a number operation what we have really done here is to give each part one inch, then another and another, subtracting in fours; but this process is more difficult to abstract than those given above.

These examples illustrate the fact that, as in multiplication, there are two different aspects of the process of division. Example i. is a process of grouping or dividing by (called Quotition). Groups of six are built up till all the children have been placed and the answer is four groups. Example ii. is a process of *sharing* (called Partition). The given number 48 is divided into a given number of parts (8). The idea of sharing is probably the easier to grasp since the child has already been involved in many experiences of sharing as a member of a family group; but the quotition aspect brings out the relationship between division and multiplication since groups are now being counted out of a total instead of groups being added to form a total as in multiplication, and paves the way for the realization that knowledge of multiplication facts can be used to deduce division facts. If $4 \times 8 = 32$ is known then $8 \times 4 = 32$ can also be known if the principle of equivalence is understood. As with the addition facts, knowledge of one relation provides the key to three others, and once this principle is grasped the number of facts to be memorized is considerable reduced (see Fig. 15).

Turning now to a consideration of how children might come to understand all the varied relationships between numbers which can be discovered by calculation, it is surely most helpful if they first meet them in concrete real-life situations, where the relationships can be discovered by operating on the materials themselves. Most children will need a great deal of experience at this level, and plenty of time to consolidate their findings, before they are presented with verbal problems and required to operate at the more abstract level. If a set of cards with simply-worded problems dealing with small quantities, and counting aids, is available the more advanced children can progress to these when they are ready. What we should recognize is that their imagination is now being called upon in a new way, both in visualizing a situation presented in words and in transferring the situation to a counting device.

Constant experience with the ladder, and any other counting aids available to the children, will help them in another way which has already been implied in the discussion. Through such activities they will discover new patterns in the number series. Probably many of them will have already discovered through counting aloud the pattern of fives, tens and elevens, perhaps too the pattern of odd and even numbers through noticing the way houses are numbered. Now, through the repetition of visual experiences while operating on the ladder they have an opportunity to discover relational patterns such as $3 + 4 = 7$, $7 - 4 = 3$, $7 - 3 = 4$; or $4 \times 5 = 20$, $5 \times 4 = 20$, $20 \div 5 = 4$, $20 \div 4 = 5$.

Some children find such discoveries very exciting and will be motivated to look for more. It is helpful now if they see the number series arranged in a variety of ways, e.g. a number square as in Fig. 12. Here the re-occurrence of the unit-interval within each tens-interval can be clearly seen, as for instance in 3, 13, 23, etc. Another interesting pattern is found if one starts at 1 and works up the diagonal through 12, 23, 34, 45, etc., or from 91 up the other diagonal. Discoveries of this kind are all helpful in bringing the child nearer to the time when the number scale will be a tool for operating at a purely abstract level.

With experience, some of these number relationships which recur frequently will become known as facts which can be used without resort to counting. Caroline was making use of one of these known facts when she replied to the question about 24 bulbs, 6 of which were tulips, with 18 straightaway when asked how many daffodils. The relationship between these three numbers was part of her mental equipment, readily available for use when needed. Children will find Arithmetic easier to deal with, and far less tedious, if they know a good many

100	90	80	70	60	50	40	30	20	10
99	89	79	69	59	49	39	29	19	9
98	88	78	68	58	48	38	28	18	8
97	87	77	67	57	47	37	27	17	7
96	86	76	66	56	46	36	26	16	6
95	85	75	65	55	45	35	25	15	5
94	84	74	64	54	44	34	24	14	4
93	83	73	63	53	43	33	23	13	3
92	82	72	62	52	42	32	22	12	2
91	81	71	61	51	41	31	21	11	1

Fig. 12. THE 100 SQUARE

	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10	11	12
4	5	6	7	8	9	10	11	12	13
5	6	7	8	9	10	11	12	13	14
6	7	8	9	10	11	12	13	14	15
7	8	9	10	11	12	13	14	15	16
8	9	10	11	12	13	14	15	16	17
9	10	11	12	13	14	15	16	17	18

Fig. 13. ADDITION TABLE TO 18

number facts as readily as this, and the second chart shown in Fig. 13, where the sequence is arranged differently, will be helpful in this connection. All the basic addition relationships are here and therefore all the subtraction relationships also. Through examining this chart some children may discover that only half the chart is necessary to contain all the facts we need (see Fig. 14). Another very useful discovery is that if you know that 2 + 5 = 7, this relationship can be used to find the answer to 12 + 5 =   , 22 + 5 =   , etc. Similarly, if you know that 3 + 2 = 5, you can deduce that 30 + 20 = 50, 5 − 3 = 2, 50 − 20 = 30, etc.

	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	
3	4	5	6	7	8	9	10		
4	5	6	7	8	9	10			
5	6	7	8	9	10				
6	7	8	9	10					
7	8	9	10						
8	9	10							
9	10								

Fig. 14. ADDITION TABLE TO 10

The table squares shown in Figs. 15-16 and the modification will be useful for learning the multiplication facts. Many of these will already be known if the children have had experience of counting in groups, but a chart of this kind will be a handy reference for some of the less familiar facts till they also become part of the memory store. Lots of exciting new patterns can be discovered in this arrangment of numbers, e.g. the table of squares running in the diagonal from 1 to 144; the patterns made by the tables of nines, tens, and elevens.

	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	18	27	36	45	54	63	72	81	90	99	108
10	10	20	30	40	50	60	70	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

Fig. 15. A TABLE SQUARE SHOWING ALL THE MULTIPLICATION AND DIVISION FACTS WITHIN THE RANGE 1 TO 144

For the few exceptionally bright children who find excitement in these kind of discoveries a set of cards might be available, including such problems as:

(i) How many sums can you make up with an answer of 10?
(ii) How many patterns can you make with 5, 2 and 7?
(iii) Four is called a square number because $2 \times 2 = 4$. How many more square numbers can you find?

What we should be looking for in all this activity is the development of what Dewey called relational thinking. Mathematical language is concerned throughout with the defining of relationships and this way of thinking about numbers and quantitative terms needs to be established from the beginning. The activities suggested here will all be helpful from this point of view if the teacher is aware of the development which is needed. In making suggestions for graded number problems at the end of this chapter this principle has been borne in mind, and they should be studied from this point of view.

1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12
	4	6	8	10	12	14	16	18	20	22	24
		9	12	15	18	21	24	27	30	33	36
			16	20	24	28	32	36	40	44	48
				25	30	35	40	45	50	55	60
					36	42	48	54	60	66	72
						49	56	63	70	77	84
							64	72	80	88	96
								81	90	99	108
									100	110	120
										121	132
											144

Fig. 16. A MODIFIED FORM OF THE TABLE SQUARE FOR USE WHEN A FACT SUCH AS 7 × 6 = 42 ALSO GIVES 6 × 7 = 42 IS KNOWN

My own experience suggests that very few infant-aged children are ready for the kind of experience we have been discussing in the last few pages. I have, however, come across a few who like Linda find the language of mathematics exciting and challenging at a much younger age than most children. If charts of the kind described, number puzzles and problems are available such children will enjoy studying them and collecting and recording interesting discoveries. These children's needs should be catered for, but most children of this age are not interested and not ready for the level of abstract thinking involved. There seems no good reason for trying to hurry (or should one say harry?) children into this level of operation, as they pick up the facts very quickly when they are ready. In fact, the writer would want to go much further than this and say that there are very good reasons for not trying to press children on to levels of experience for which they are not ready mentally or experientially. In so doing we are likely to cause bewilderment and frustration and so to rob them of the joy of making these discoveries at a later stage when they are ready.

We now come to a consideration of the timing of written-reckoning and might begin by asking, when does it become necessary to resort to this? The answer, of course, is that there is no need to write down a calculation which we can perfectly well do mentally. It only becomes necessary when the steps involved are more than we can remember; in other words when we are operating with large numbers which must be dealt with by stages. Children should surely not be required to deal with calculations involving large numbers until they are thoroughly familiar with the operations involved through handling small quantities, first at the concrete level, then one stage removed from this with a counting-aid such as the ladder, until finally they dispense with this, finding it no longer necessary except,when they meet with a number relation which they have forgotten, or never knew very well because they haven't met it many times.

There are, however, children whose auditory memory is not strong and who remember more readily things which have been presented visually than those which they have only heard spoken. Such children begin to make much more rapid progress once they can understand and use written statements about number operations.

When children are judged to be ready for this step it is helpful to introduce the first written statements in the form of 'writing down what we have done'. This might be a record of a shopping transaction, the results of a scoring game, a story involving numbers, or an account of a measuring activity. The matter might be recorded in pictorial form with a written statement underneath. Some children will need a good deal of help at this stage if they are to attain a real understanding of the statements they are writing and the teacher might find it helpful to build up stories and records with them using a flannelgraph board. Once they got hold of the idea many children enjoy making up number stories and in Miss Adams' book there are some delightful examples of children's work with a full discussion of this approach which is well worth reading.

Little problem cards made by the teacher suggesting operations to be undertaken and recorded will be useful at this stage, and if carefully designed can help children to acquire an understanding of the signs used in number equations, and facility in writing their statements in the most economical way. Some children find little difficulty with the language of numbers but many do, and the teacher needs to be aware of those who find it difficult to interpret the symbols they are using. If, at this stage, children receive the help they need many of the later difficulties in arithmetic may be avoided. For example, we noticed in Chapter I that Caroline did not understand the function of the minus sign. If she had been taught to think of it as a direction sign she would not have written her sum as $\begin{array}{l} 06\,- \\ \underline{24} \end{array}$ but as $\begin{array}{r} 06 \\ \underline{-24}. \end{array}$ Many odd things are done with mathematical signs due to a lack of appreciation of their function, but a child who is ready to learn to use the signs is also ready to understand the meaning they carry.

It is obvious that a fair degree of reading skill is necessary for this kind of work but I am assuming that children who are ready for this level of number activity are already well advanced in reading and writing, since in my experience readiness for reading comes much earlier than readiness for number operations because of the abstract nature of number language.

When children have had plenty of experience of this kind and show that they understand what they are doing it is useful to proceed to write down operations undertaken on the ladder or number tract. This, as we have already seen, will help them to take a step away from concrete situations towards the stage when they are able to deal with numbers as such without the use of any concrete aid.

By this time we shall have needed to extend our ladder, perhaps to a hundred, and with this as a basis children will not find it difficult to write down operations involving double-figured numbers, including those involving the 0.

There comes a point, however, when children need to understand the function of this symbol if they are to avoid misunderstanding when they want to operate without the ladder. For most children this will not be until they have reached the Junior School, but some suggestions regarding this are included here for the sake of the exceptionally bright children who may benefit from this bit of experience earlier than most.

Again, the best way to approach an understanding is to see the matter first at the concrete level, and an abacus of some kind is necessary for this. At first the children will play with it, putting on single numbers and perhaps writing down the number represented on the abacus. Here is a concrete presentation of the principle of place value and the meaning of the zero sign becomes obvious in time. The abacus can now be used for addition and subtraction, and writing down consists in 'writing what we have done on the abacus'. Many of the difficulties children commonly experience with 0 may be avoided if they are given an opportunity to learn first in a concrete situation and taken back to it if they do get into difficulties subsequently.

The latter half of this chapter has taken us into a discussion of operations which are beyond the attainment of most Infant-aged children. It is included to indicate the direction which progress in calculation will take. The teacher should always be aware of the more advanced stages of learning for which her children are preparing themselves. This awareness will help her to evaluate their activities and to ensure that their present experience is providing a sure foundation for what is to come.

As in the last chapter we shall conclude with some suggestions for activities which will aid the progress we have outlined. Again we are regarding as most important the using of everyday experiences and natural play materials which provoke grouping, comparing, sharing, ordering, etc., and as a supplement to these but never in place of them, games and apparatus specifically designed for number education. Where the teacher is fully alive to the potentialities for learning to number in the former, and exploits these to the full, she is unlikely to need very much of the latter. There is no suggestion here as to what is most appropriate for classes of children of different ages, since as teachers know very well there may be a considerable spread of mental age, ability and experience in a class of children whose chronological ages may only vary by six months. Equally teachers who have taught in more than one school will know how the age of readiness for number experience can vary from one neighbourhood to another.

SOME SUGGESTIONS FOR THE TEACHER

1. *Opportunities for sorting.*

   Collections of shells, leaves, buttons, coloured beads, mosaic shapes, etc.

   Finding what you need for a building construction, woodwork, jig-saws, pattern-building.

2. *Opportunities for matching.*
   Knives and forks, buckets and spades, bats and balls, paint pots and brushes, brushes and combs, etc.
   Fitting toys, jig-saws, mosaics, building bricks.

3. *Opportunities for ordering.*
   Tillich bricks, graded rings on stick, nest of boxes or cups, building bricks, beads of graded size, measuring activities.

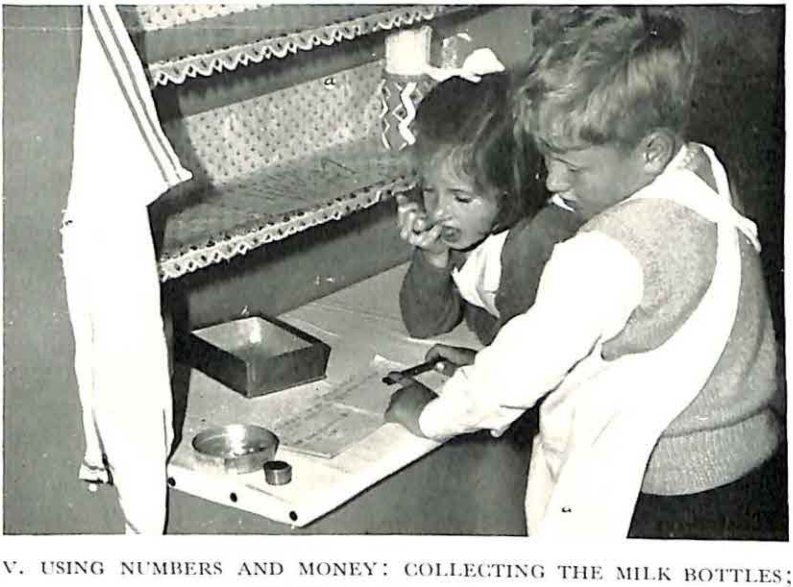
4. *Opportunities for grouping.*
   Laying tables, putting out classroom materials, Wendy house play, groups for play activities, play with sand and clay, building operations.

5. *Opportunities of sharing.*
   Sweets, biscuits at meal times, play materials which have to be shared.

6. *Opportunities for counting in ones and in groups.*
   Number of bottles of milk required.
   Checking classroom materials—paint brushes, scissors, etc.
   When looking at pictures—in books, including those children have made.
   Number rhymes and jingles.
   Folk and fairy tales.
   Buying from the school shop, or classroom play shops.
   Friezes of various kinds—a row of houses, groups of soldiers, Noah's Ark, the Three Bears.
   Measuring activities.

7. *Games for counting and comparing.*
   Bead-threading, tower-building, making collections.
   Skittles, fishing, balls and a basket (with concrete tally of some kind), see page 135.
   Dice games, e.g., the board variety usually called 'Home Games' which can be designed in various ways according to the needs of the children; the floor variety such as a ladder painted on the corridor floor.
   Measuring and shopping activities. Cooking.

8. *School Shops.*
   These can be organized for use at many different levels and should be varied according to need and also to maintain interest, e.g.,



V. USING NUMBERS AND MONEY: COLLECTING THE MILK BOTTLES; COUNTING THE TAKINGS.

VI. USING SHAPES: MANGLING; MACHINING.

i. Toyshop with all goods marked in multiples of pennies.
ii. Sweet shop with a variety of prices including shillings and where money available includes pennies, threepenny pieces, sixpences and shillings. Labels might include such varieties as 3 a 1d., 1/– for ½ dozen, 5d. a ¼ lb., 1d. each or 6 for 4d.
iii. Greengrocer's shop with a variety of prices, e.g. 4 for 6d., 1/6 a lb., 3/6 a box, 2/9 a dozen. The children might be given 10/– to spend, including 2/6 and 2/– pieces.
iv. Draper's shop with prices such as 4d. a doz., 4/3 a yard, 7/6 each.
v. Cafe with menu cards and so much to spend on a meal for the family.
vi. Post Office—stamps of different values, postal orders, stamped postcards and envelopes, letter cards and air-mail letters.

If these are carefully planned with the needs of the children in mind they can provide valuable experience of number values. In one school I know, a variety of shops are arranged in the corridors. In this way they are freely available to all the children and they are able to use those they find most interesting. In this school also real money is used which seems to add to the value of the activity. This can provide a real incentive for checking and counting. Those who wish to be shopkeepers on any day have to apply first of all to the Bank for a till containing a certain amount of money for change. This must first be checked before the shop is opened and the same amount handed in at the end which requires some time for operations after the shops are closed. Those children who wish to shop on any day apply to their teacher for a purse containing a certain amount of money which must be checked before they leave the classroom. At closing time they return their goods to the appropriate shop, receive back the money they paid for them, and after checking the amount in their purses, hand them back to the teacher.

In some schools teachers may not feel it is possible to organize affairs on this scale but might like to have one real shop where the children can buy biscuits and chocolate to eat with their milk.

A school bus where tickets have to be bought for journeys to different places; and a place where puppet shows and other entertainments can be laid on for which an entrance fee is charged also suggest themselves in this context.

9. *Counting aids available for the children.*

A string of 100 beads stretched along one side of the room.

Collections of beads from 1 to 10 hanging from wall.

Ladders, boxes of counters, bead frames of various kinds, number blocks, a set of Tillich bricks, a Montessori stair.

Friezes with objects to count, e.g. a street of houses, rows of marching soldiers, a park full of cars.

Friezes which tell a number story, e.g. the Three Bears, the Ten Little Nigger Boys, Noah's Ark.

A calendar, foot-rulers, tape-measures, marked measuring jugs, weights, a clock.

Number squares of different patterns.

Books about numbers in the book corner.

An abacus for the brighter children.

Structural material of the kind devised by C. Stern, Cuisenaire, and others.

10. *Graded Number Problems.*

A set available for the more advanced children graded in difficulty from those involving quantities small enough for the answer to be discovered without resort to written calculation to those involving larger quantities and more complicated problems for the brighter children.

11. *Opportunities for the exceptionally bright children.*

Number puzzles.

Games for memorizing number facts, e.g. lotto, darts (those adhering by suction!), number snap, monopoly.

Railway time-tables, an AA guide, maps of different kinds, catalogues, books containing statistics which are of interest to children, holiday guides, etc.

These can be used by the children for pursuing their own interests or for inquiries suggested by their teacher. I have also seen them used by a group of children for setting problems for each other to solve.

D. E. Smith 'Number Stories of Long Ago,' for the interesting puzzles it includes.

A NUMBER READINESS TEST

1. Put out two unequal-sized groups (e.g. 10 and 13) of sweets in front of the child and ask him which pile he would rather have. Then ask 'Why did you choose this pile?' Repeat with smaller groups (e.g. 6 and 4) if the child fails on first test.

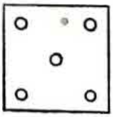
2. Present child with two glass beakers of equal size and shape, one containing 20 small beads or balls. Ask him to put the same number of beads in the empty beaker and if his method is not clear ask him how he knew how many to put into his beaker.

3. Have ready a jug containing lemonade or some coloured liquid, and four glass beakers: two of the same size and shape, one taller and narrower, one shorter and wider. Place the two identical beakers in front of the child and after pouring some of the lemonade into one, ask him to pour the same amount in the other beaker. When he has finished ask 'Have the two beakers got the same amount of lemonade in them?' and then 'How do you know?' Then take the taller beaker and ask the child to empty the lemonade from his beaker into the new one. Place this when filled alongside the other beaker which still holds lemonade and ask, 'Which lot of lemonade would you like to drink?' By questioning discover the reasoning used to arrive at his answer.

4. Place a large pile of sweets (about 30) in front of the child and ask him to share them out with you so that both of you have the same number of sweets. Watch how he solves the problem (by rough handfuls, by building up in ones to form two equal groups, etc.). Then take the pile he has given you and spread them out on the table so that they occupy a much larger surface than the child's pile. Now ask him 'Which lot of sweets would you rather have?' and by questioning try to discover his mode of reasoning.

5. Have ready a box of counters and a series of cards with counters painted on them in the following manner:

a)



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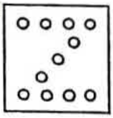
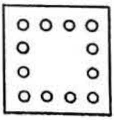


c)



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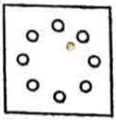
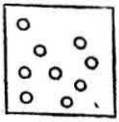


e)



Present each card in turn and say 'Put out as many counters as there are on the card.' Notice the way he achieves the correspondence (pattern-copying, counting, etc.).

6. Put out counters on the table in two dissimilar groups (e.g. 3 and 4.) Ask 'Where are there more counters?' Follow this, according to response to first question with either 'How do you know?' or 'How many more are there here?' pointing to the group the child has chosen. Repeat using groups of 8 and 7, 11 and 13.

7. (*a*) Give the child 7 pennies and ask 'How many pennies have you got?' Give him 5 more pennies and ask 'How many have you got now?' Notice the way he arrives at his answer (counting the whole group, counting on from 7, etc.).

   (*b*) Give the child 8 small bricks and ask him to arrange them in a straight line. Then ask 'How many bricks are there?' Now add 9 more to the row yourself and then ask 'How many bricks are there altogether now?' (Leave a small space between your row and his so that counting-on is a possible method of obtaining the answer.)

   (*c*) Ask the child to put 3 sweets into a paper bag and then to add 4 more. Ask him 'How many sweets are there in the bag now?' Repeat with 6 and 5 sweets.

8. Have ready the cards used in test No. 5 and a set of cards with the numerals printed on them, 1 to 12 are needed. Ask the child to find the numbers which go with each card.

9. Ask the child the following questions:

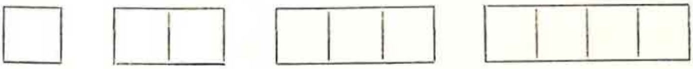
   There are 4 cars, how many wheels?
   ,, ,, 6 children, ,, ,, legs?
   ,, ,, 6 ,, ,, ,, toes?
   ,, ,, 3 babies, ,, ,, fingers?
   ,, ,, 5 chairs, ,, ,, legs?

10. (*a*) Give the child a bag of 12 sweets and say 'I want you to share these out between 3 children so that each has the same number. Notice the method used (three rough groups, building up three equal piles in ones, etc.).

    (*b*) Repeat using 30 sweets and 4 children. Notice method used and what child does with the remainder.

(*c*) Present child with a piece of ribbon or strip of paper 16 inches long and ask him to cut it into four equal pieces. Have a tape-measure on the table so that the child can use it if he wants to.

11. Present the child with a series of ten Tillich bricks or set of ten cardboard strips similarly graded in size and marked off in unit lengths, viz.:



Ask the child to arrange them in order and when he has finished point to the first in the series and ask 'Why did you put this one here?' Repeat question pointing to the last in the series, and then to one in the middle.

12. Arrange the series to form a 'stair' series if the child has not already done this, and ask the following questions:
    (*a*) Pointing to the sixth stair 'Which stair am I on now?' Repeat with finger on first and eighth stairs.
    (*b*) Put finger on bottom stair and ask 'How many stairs will my finger have to touch before it reaches the top of the stairs?'
    (*c*) Put finger on bottom stair and ask 'How many stairs will my finger have to touch to reach the eighth stair?'

13. Repeat experiment in tests 11 and 12 with a series of unmarked strips of cardboard. Notice manner of response (whether attention is directed to size differences or each strip given a numerical value).

14. Have ready a series of picture friezes: a row of houses, a line of cars one behind the other, a line of cloakroom pegs; also the set of numerals used in test No. 8.
    (*a*) Present frieze of houses and say 'I want you to give each house a number.' Then ask, pointing to the first house, 'Why have you given this house this number?' Repeat question pointing to fifth and eighth houses.
    (*b*) Present frieze of cars and give child card with number 5 on it and say 'Which car should have this number?' Ask 'Which number will be needed for the last car?' and let him find and place it on the car. Repeat, using sometimes the first and sometimes the second type of question till all the cars are numbered.

(*c*) Present the frieze of clothes pegs and tell the child 'Your peg is No. 7. Find the number and put it on your peg.' Repeat using the names of other children he knows till several of the pegs are numbered. Then say 'Now tell me the names of the pegs which are not being used by any children.'

Comments on the Test Items

It is intended that these tests shall be administered to individual children in an informal setting, perhaps while the rest of the class are engaged in their freely chosen activities. Three or four might be administered on one occasion, and the others subsequently.

The purpose of giving the tests is to find out how the child approaches situations which for their successful solution require an understanding of the basic number concepts and skills. Piaget divides the responses he obtained in similar tests into three categories, according to the level of response, viz. (*a*) pre-operational, (*b*) intuitive, (*c*) operational, and this method of assessment may be found helpful. Any response which suggests that the child is not seeing the situation as a quantitative one, involving counting, will go into the pre-operational category, and any response in which the child uses counting or measuring operations appropriately, and for the successful solution of the problem, goes into the operational category. It will be found that these two levels of response are easy to distinguish but many children show an unevenness of response, some things understood and others not, which suggests an intermediate stage in development when the concepts are being acquired but are not yet stabilized. E.g. at one moment the child will establish the size of a group by counting and at another he will make global judgments of quantity based on spatial consideration. A series of tests of this kind is particularly helpful in ascertaining those children who, though on some occasions look as though they understand the quantitative aspects of a situation, when observed systematically reveal how uncertain they are as compared with children who have reached the operational level and are ready to proceed to the more abstract problem.

These tests have already been used by a small group of Infant Teachers in one area who have found them helpful in distinguishing the level at which children are thinking. However, we have found it necessary to question the children about why they have done things in certain ways in order to understand how they are looking at the situation. Sometimes the child's own spontaneous comment will give the information needed as in the case of the little boy numbering houses who suddenly turned them into jockeys going for a race. More often this is not immediately clear as we found when asking children why they had chosen one pile of sweets rather than the other in Test No. 1. One child who had chosen the smaller pile explained that it would be greedy to choose the larger one.

It has been thought helpful to include some guidance as to what to look for in each of the tests, but these remarks should be regarded as suggestive only. It would not be possible to cover all the eventualities which can occur during testing and what follows is intended merely as a guide to the kind of concepts being examined in each test item.

The Group Concept.

No. 1. Children who have learned to think of groups of discrete items as comprising sets of different sizes whose numerical value can be established by counting are likely to use this method in solving these problems, particularly with the larger sized groups. Children who have not reached this stage are likely to base their choices on the size of space occupied by the various groups of sweets, responding to the question by some such reply as 'Because it's bigger' or 'Because it's more'.

No. 2 will give information of the same kind. The child who has attained the group concept will recognize that he must first count the number of beads in the filled beaker before he can know how many to put in the empty one. The child who has not reached this stage is likely to try and fill the empty beaker with beads until they reach the same level as those in the filled beaker, i.e. so that they look the same. Some children when asked how they knew how many to put in will now recognize the quantitative aspect of the situation and proceed to count the beads in the two beakers. This would be an example of the intermediate level response.

The Concept of Conservation of Quantity.

Nos. 3 and 4 will show those children who understand that a quantity (whether continuous as in No. 3, or discontinuous as in No. 4) always retains its quantitative value no matter what other changes may occur. The child who does not understand this is likely to choose the beaker which looks to him as though it contained the larger amount of lemonade and similarly the group of sweets which occupies the larger space. Usually these children will divide the group of sweets into two rough piles, perhaps making one or two adjustments till he is satisfied that they look the same. On the other hand the child who has attained the concept of conservation of quantity will recognize that the original quantity of lemonade or sweets is not altered because it now occupies a larger space. These children are likely to divide the sweets in test No. 4, by building up in ones to form two equal groups.

No. 5 will give further information about the attainment of the group concept. Some children feel no necessity to count the number of counters on any of the cards but concentrate on reproducing the pattern, often taking great care to reproduce the details of pattern and colour with great exactness. The more advanced children may also deal with this test in the same way, and it cannot therefore be regarded as useful diagnostically unless it is used in conjunction with other tests in the series. Some children will count to establish the size of the sets before putting out any counters, particularly in the later models in the series.

Enumeration

No. 6 is helpful in distinguishing those children who understand how to count. When the larger groups are involved correct answers can only be given after a counting operation and the difference between children who understand this and those who do not is very obvious.

Counting-on

No. 7. These situations can be dealt with either by counting the second group from one, or by counting on from the position in the number series already reached. The last item in this series is likely to be most informative since the group is hidden in the bag, unless the child takes them out to count before answering the question.

Knowledge of the Numerals

No. 8 is included to show whether the child has learned to associate the correct value with each numeral.

Ability to Count in Groups

No. 9. These items will readily be solved by children who have learned to count in groups. As has been noted already this is a useful skill and a helpful basis for later work in multiplication.

Ability to Share

No. 10. The first two items in this series can only be solved if the child has attained the group concept and understands how to share a group of discrete items. Children who have not reached this stage are likely to be satisfied to divide the sweets into 3 and 4 rough groups respectively. As has been noted already in discussing division the last item is likely to be dealt with by folding the ribbon in half and then half again. Those experienced in measuring may use the tape-measure provided.

The Concept of Ordinal Number

So far in the test series all the counting has been used to establish the size of groups, i.e. cardinal number. The last three in the series test the child's appreciation of quantitative order.

No. 11. presents a situation in which the size relationship between the bricks can be used as a principle for ordering the series. Many children interpret the instruction 'arrange them in order' as a request to put the bricks tidy and arrange them in a long line or a tidy group, without respect to size difference. The more advanced children are likely to notice the size difference and use this as a basis for ordering, though some find difficulty with the larger bricks where the difference is not so easy to distinguish unless the child builds a stair series or counts the unit lengths on the bricks.

No. 12 will show whether the child understands the ordinal or positional function of number.

No. 13 provides further evidence about this. Children who have played a good deal with Tillich bricks and used them for ordering operations will often refer to them by their number names as represented by the number of units marked on them. In this test it has been found that some children will transfer this method of thinking to unmarked strips.

No. 14 presents situations of the kind which frequently occur in everyday life where numbers are used to indicate position in a series and where the cardinal aspect of number is not present as it is in the Tillich bricks. It will be found that these tests clearly differentiate between the more advanced and less advanced children.

CHAPTER NINE

AN EXPLORATION OF SPACE

SPACE is in the news these days and while politicians are debating control of outer space and scientists devoting their energies to the perfecting of moon-rockets, earth satellites and the like, their children are quietly accommodating themselves to living in a world fundamentally different from any world which their teachers dreamed of in their childhood.

Space-men are competing with cowboys and Indians in children's comics, films and TV programmes, and that they have become the heroes of the mid-twentieth century child is strongly suggested by the frequency of their occurrence in the pictures they make, the stories they write, and the plays they act. The exploits of man-made satellites and rockets are followed with interest and many a young man of seven years old has a vocabulary of spatial terms exceeding that of his teacher.

Nevertheless even were this not so space is one of the basic human experiences with which every child of every age has had to come to terms, and we must begin this chapter, as we have the preceding ones, with a consideration of the little child's earliest experiences of space.

The first space experienced by the human organism is the warm watery space in which he lives as a little aquatic organism for the first nine months of his existence. To emerge for the first time into a different kind of space—dry, rarer and colder—must be a considerable shock, and requires immediate accommodations if he is to survive at all. New ways of breathing and feeding have to be learnt, new positions are taken up, and new sensations of many kinds are experienced providing information about the new kind of space in which he will be living from now on.

During the first few months of life the infant is mainly concerned with this kind of learning and only very gradually does he become aware of objects in the space around him—the woolly blanket which covers him, the soft pillow on which his head rests, the nappy which is tied on to his body, the breast or bottle from which he feeds, the arms which hold him, the milk which he takes in, the waste products which he gives out, the warm water in which he is bathed. All these experiences are spatially-related to him in different ways, a feature which aids him in the business of differentiating them from each other, and it is not long before he begins to show by bodily responses to these different experiences that he is coming to terms with the space outside himself.

But these experiences also enable him to become aware of his own inner space, to notice events inside his own skin. Experiences of hunger, pain, satiation, comfortableness, of lying down in the cot, of being lifted out of the cot, of being held upright in mother's arms, of being rocked to sleep—all these are experienced as changes in the child's inner space, reported to the central nervous system as interoceptive, proprioceptive and tactile sensations—and responded to by accommodating bodily movements which in turn yield further proprioceptive sensations. Because all the information yielded by sensori-motor activity is stored within the brain the little baby is gradually building up a map of his own body in which all his inward parts are spatially-related to each other. This body-image, as we noticed in Chapter IV, is essential to his survival. It enables him to make the spatial adjustments which are a necessary condition of homeostasis (the maintenance of stability in biological functioning).

It is in these early spatial adjustments that the child achieves his first spatial experiences and herein lie the roots of spatial knowledge. From what we know of the diffuse nature of sensory experience in the first few weeks of life we assume that there is no real differentiation of outer from inner space experience—it is only body experience of a very generalized kind. But in time, probably due to experiences of his mother, that which is outer and separate is sensed as distinct from that which is inner and contained. We can infer this development from behavioral changes: the attention he begins to pay to movable objects in the space around him, the efforts he makes to reach out for them, and the movement skills he repeats; all of which suggests that he is becoming knowledgeable about his outer as well as his inner space.

The psychological relationship between the outer world and the inner world of the young child has been explored by Prof. Meredith in a paper given to the Chichester Conference of the World Federation for Mental Health in 1952.[1]

He begins with the obvious fact that the child has a body which has a certain size and shape, and certain capacities for movement. This body which is held by gravity to the earth's surface is surrounded by material bodies and living organisms.

> These bodies are all extended in space and many of them capable of movement. In any case the child must move in order to reach them. It is this fact which gives rise to the problem of spatial development in his mentality. *Space is not a self-sufficient entity but one kind of abstraction from the series of events which constitute the development of the human being.*

Moreover these outer events are not only extended in space but occur and re-occur in a temporal series. Some of these events such as meals, baths, being put to bed and being dressed recur with fairly rhythmic regularity and 'provide a first time-scale against which the more irregularly occurring episodes are placed and measured'. The child has to learn to fit into the man-made framework of time in which his society operates, which gives rise to 'the problem of temporal development, i.e. the articulation of the child's mental time and of his temporal language so as to engage in the gears of his environment'.

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[1] Meredith, G. P., 'The Space, Time, Language and Intellect of the young Child', in *Mental Health and Infant Development,* ed. K. Soddy.

In addition, this outer world is full of significant sounds which we call language, words which people use to define these events. Thus there are naming words, or nouns, to associate with the shapes he explores; verbs to express the interactions between these shapes which occur in time; and later adverbs and prepositions to define the events more concisely, e.g. there is *Mother* and there is *biscuit*; there is mother who *gives* Johnny the biscuit; and later there is Johnny who sees the biscuit *over there* and wants Mother to give it to him *now*.

How then does the child come to terms with this outer world? How does he build up his inner world? Here we have to consider both the activity of the sensory and motor apparatus of the body and the activity of the central nervous system in which past experience is stored in the form of active memory-traces which blend in a dynamic way with current sensory impressions to pattern events.

As we have already noticed there are three groups of sense organs contributing information. There are the exteroceptors on the surface of the body—eyes, ears, nose, tongue and skin. Together they provide 'a more or less constant spatial framework' which 'serves as the focus for the *perspective of the spatial framework*. The introjections of the body-surface, and of the environment, on the screen of the inner world are related in a way which lays the foundation of the *inner space*.' Secondly there are the interoceptors in the lining of all internal body organs which report on the general tone of the inner world, 'a tone which may be mellow or harsh, calm or restless, exuberant or painful, a tone which saturates everything on the screen whether objective or subjective in origin'. Thirdly, there are 'the proprioceptors by which the muscles, tendons, joints and semi-circular canals report to the brain on their actions and on the spatial orientation of the body'. For this reason, as Professor Meredith points out, the latter are 'of particular importance in relation to the space of the inner world'.

> The *location* of the body in the environment, the *anatomy* of the body itself and the *receptor-connections* of the surface, the viscera and the motor organs with the cortex, between them provide certain guides to the gross structure of the inner space. Clues to the detailed structure . . . are to be found in *physical movement* and in social behaviour. . . .
>
> The infant builds up his capacity for perceiving the spatial forms of objects by a slow process of correlations between visual and tactile impression and proprioceptive sensations from the eye-muscles, the hands, the tongue and the lips. As he starts to crawl, to walk, to run and to climb he builds up the larger space of distance, direction and elevation, and comes to translate the visually foreshortened perspective space of his environment into the Euclidean space of bodily movement. Thus *movement must be regarded a principal generator of the inner space.*

If this is so then it follows that the young child must be allowed freedom of movement within the limits of safety and also that he must be given the language in which to define his movement experience. Professor Meredith suggests that thinking may be described as the introjection of bodily actions and goes on to say that:

> in the process of introjection language plays a crucial rôle. It enables actions to be given symbolic representation in the inner world, thus setting them free from the limitations of specific imagery. In so far as the child is helped by his parents to associate his movements with precise verbs, adverbs and prepositions, the introjections of these movements become precise *mental operations.*
>
> Bodily actions include not only the whole range of locomotor actions of limbs and trunk but also the extremely varied actions of the hands, in manipulating all that the environment offers—materials hard and soft, plastic and elastic, greasy and sticky, clean and dirty, inanimate, living and human, powders and liquids, froths and jellies, winds and flames, in all manner of inter-relations and interactions—the fitting of lids on saucepans, of buttons into button-holes, the turning of handles and the pressing of switches, the folding of paper, the breaking of sticks, the rolling of cloth, the pouring of liquids—one could continue the list indefinitely. The quantity of self-pedagogy in which the child indulges in the early years is prodigious. He establishes all the experiential foundations of physics, chemistry, biology,

> sociology and mathematics. . . . He is hungry for the knowledge of the properties of matter. And thus he develops a labyrinth of fine structure in his inner space.

The significance of bodily activity in the development of spatial concepts has been noted by other writers, notably Sherrington, Cassirer, and Piaget.

The first bodily-experienced space is defined by Cassirer[1] as *organic space*, and, as already mentioned in chapter 2, this kind of spatial experience is the only kind open to the lower organisms, but it makes possible, even in these, a power of differentiation and orientation which results in behaviour in ants, bees and migratory birds, which is a source of wonder to man.

The instinctive behaviour of these creatures which is so beautifully described by Fabre in *The Wonders of Instinct* is, however, stereotyped in form, and not susceptible to the range of modifications open to man due to his more highly-developed nervous system and increased capacity to learn from experience. This is clearly illustrated in the following account take from Ferton's study of the solitary bee.

> A small solitary-bee uses the house of a certain species of snail as a place to lay her eggs. After the eggs are laid in the snail's shell, the bee makes a hole 6–7 cm. deep in the earth, and places the shell in this hole. When Ferton removed such a shell from the hole in which the bee had just slipped it, the insect continued its work, filling the hole, smoothing over the earth, and otherwise finishing the task exactly as if everything were normal.[2]

Because of his more highly-developed nervous system man experiences *perceptual space*. By means which psychologists and physiologists are only beginning to understand, sense data is related and patterned in the central nervous system and yields information in a form which allows us to give it meaning. These meanings, which are no more than perceived relationships in inner and outer space, are being added to daily due to the little child's bodily activity. It is, to use Cassirer's term, *a space of action and the action is centred around immediate practical needs and interests*, and it can be compared with primitive man's conception of space of which Werner writes:

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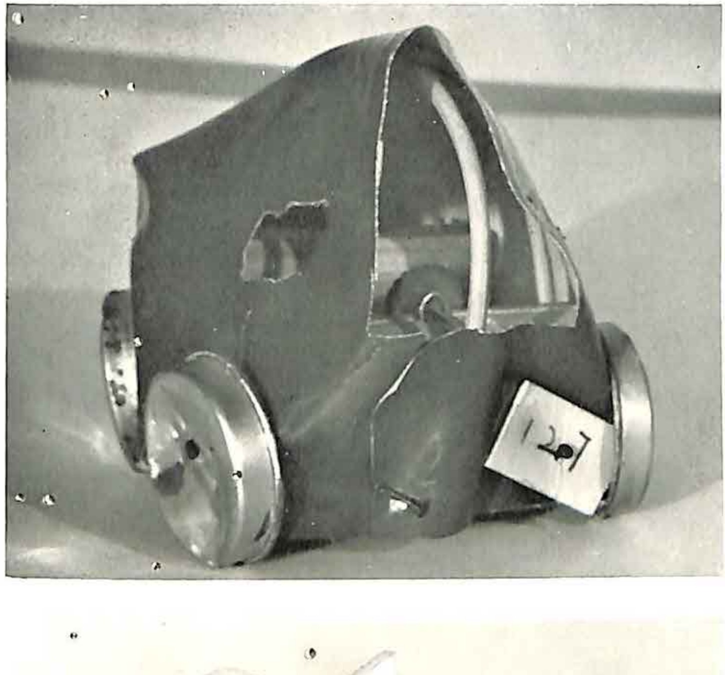
[1] Cassirer, E., *The Philosophy of Symbolic Forms*. Vol. I.

[2] Quotation from Werner, H., *Comparative Psychology of Mental Development*.

> So far as the primitive man carries out technical activities in space, so far as he measures distances, steers his canoe, hurls his spear at a certain target, and so on, his space as a field of action, as a pragmatic space, does not differ in its structure from our own. But when primitive man makes this space a subject of representation and of reflective thought, there arises a specifically primordial idea differing very radically from any intellectualized version. The idea of space, for primitive man, even when systematized, is syncretically bound up with the subject. It is a notion much more affective and concrete than the abstract space of the man of advanced culture. . . . It is not so much objective, measurable, and abstract in character. It exhibits egocentric or anthropomorphic characteristics, and is physiognomic-dynamic, rooted in the concrete and substantial.

If we look at the first efforts of young children at representational drawings of people and houses we find features which accord with this description. They can tell us a great deal about their grasp of space relations.

With the advent of language there comes the possibility of defining these perceptual sensori-motor structures which have been built up. Come, go away, sit, down, and up are some of the first spatial words to be understood, and the movement quality in these words is significant. They are first understood in terms of his own actions and the actions of others which are like his own. With the increased mobility which comes with the power to walk, and the accompanying increase in manual dexterity, opportunities for exploring space and the shape of things in space are enormously enlarged. Even before this the little child has gained a good deal of information by exploring shapes with his fingers, mouth, and eyes. Thus roundness is learned in terms of the shape of the hands to hold it, likewise thinness, flatness, smallness and bigness. Thus also nearness,



VII. MAKING SHAPES: A BUBBLE CAR; A WOOD TRUCK.

VIII. MACHINES—HOW DO THEY WORK?: A RAMP; A PULLEY.

separateness, bigness, smallness, softness and hardness in terms of movements needed to reach and grasp objects. As Piaget says:

> Abstraction of shapes is not carried out solely on the basis of objects perceived as such but based to a far greater extent on *actions* which enable objects to be built up in terms of their spatial structures.[1]

In the symposium on Child Development, Piaget was at pains to emphasize that there are in the development of the child two kinds of experience, always more or less intermixed, but which give rise to two kinds of knowledge or two kinds of methods of acquiring knowledge. 'There is first the experience which I will call "physical experience" or "experience derived from the physical environment" which is experience in the accepted sense of the term.' This kind of experience 'furnishes knowledge taken from objects and gives rise to what one might call an abstraction derived from the object'. For example:

> it is by experimenting with objects that the child, before he is able to speak, at the sensori-motor level, notices that there are heavy objects and light objects. Then this knowledge gives rise to associations. He discovers that heavy objects are generally larger and larger objects are generally heavier. . . . This type of *knowledge is acquired by abstractions derived from the object.*[2]

Incidentally, it is this kind of experience which gives rise to the size-weight illusion which we noticed in Linda (see p. 115).

> There is, however, a second kind of experience which I should like to stress. Experimentation here is also carried out on objects and can be carried out *only* by manipulating objects. But in this case the knowledge is not drawn from the object, and the abstraction does not derive from it. *The abstraction is derived from the actions themselves* which are performed on the object, which is something quite different.

For example, Piaget found that in an experiment which consisted in comparing the lengths of two lines, one vertical and the other inclined at different angles; the comparison was much easier for the child than the adult.

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[1] Piaget, J., *The Child's Conception of Space.*

[2] See *Discussions on Child Development*, Vol. II.

> The child is able to make much more precise comparisons than the adult. Why is this? It is because after a certain age *a perceptive space is organized according to systems of co-ordinates which presuppose patterns based on co-ordination of actions*. A small child does not yet have this system. He sees things directly, independently of their orientation in space.

Later in the discussion Piaget states his view that:

> reasoning is the product of *internalized action*. Reasoning is the result of a series of operations which were first actions; it is simply a reflective level of internalization and of symbolization, permitting reasoning, but originating at the sensori-motor level, before the development of language and inner thought. . . . . In the fields of logic and mathematics we are able to reason and make deductions without reference to experience. It is not the same as regards physical experience. That is the great mystery of mathematical learning: you are very soon able to do without experiment and to carry out voluntary reasoning. But you are able to do that just because it is a different type of experience from the physical experience. You have *experimented with your own actions* (e.g. in ordering, comparing, etc.), and once they are co-ordinated you no longer need the objects: you can continue symbolically, internally. There is a certain logical mathematical type of deduction which a child of seven has mastered, whereas physical deduction seems more difficult.

The distinctions made here by Piaget are very important and will repay careful study. It links up with a number of discussions in this book as the reader will realize.

By the time they enter school little children are fairly sophisticated with regard to space. They are able to manage their own bodies pretty well for standing, walking, running, hopping and skipping about in space. They know how to sit down on the floor, on a chair, and at a table. They know how to reach up to a shelf, to crawl under the table, to turn the handle of the door, to push open the cupboard, to fasten the gates together, to walk upstairs, to climb into the bath, to get in between the bedclothes. They know how to hold the cup and jug by the handles, to handle spoons, forks, chalks, and maybe paintbrushes and scissors. They know that there is a difference between arranging their bricks alongside each other and pushing them along the floor, and building them upwards into space in the form of a tower. Home is a familiar space now and the little five-year-old knows in which direction to go if he wants the lavatory, where Mummy cooks the dinner and does the washing, where they have their meals, which is Daddy's chair, and which is his toy-cupboard. This knowledge has given him an increasing independence of adults and he is already showing considerable motor and manipulative skills.

In the Nursery and Infant School it is our business to provide for the extension of this kind of knowledge and skill, and a good deal of thought needs to be given to the physical environment from this point of view. It must allow space for movement since it is only through movement that children can learn to conquer space, and there must be a wide variety of things in this physical space which provoke inquiry and experiment and make possible the discovery of new spatial relationships and the development of new movement skills in relation to them. Perhaps most important of all is the presence of a teacher whose attitude towards them makes it possible for the children to be adventurous and experimental in their play. The over-anxious teacher who is fearful lest the children shall hurt either themselves or each other, and who finds it necessary to interfere with their play at frequent intervals, will inhibit learning and may communicate her anxieties to such an extent that the children become timid into their approach to materials. Naturally, one would not wish teachers to be irresponsible over the welfare of the children entrusted to them, and it is important for each one to plan an environment in which she can allow her children to play freely without undue risk of hurt. It is interesting to notice how very greatly teachers vary as to what they feel they can allow safely. The writer has seen little four-year-olds playing freely with play materials and tools which some teachers would not be able to allow to six-year-olds. On the whole little children seem very capable in judging what they can do safely, and gain in skill as well as confidence when they are trusted. It is in conditions of over-excitement and tension that accidents happen, and the experenced teacher knows how to recognize these situations when they arise and deflect attention to something else when necessary.

The developments in spatial knowledge to be expected during the Infant School years might be thought of in this way. There are first of all the new discoveries to be made—the differentiation of the many varied shapes which physical bodies can assume, a growing appreciation of the patterns in space which are made by moving bodies, the relation between one's own body and a spatial environment—expanding in size and complexity, and the patterning of this expanding space until one comes to feel thoroughly at home in it. And accompanying these developments there is the acquiring of new words which can be used to talk about, define and classify spatial experience.

Let us now consider from this point of view some of the experiences which are available to most Infant School children these days. Last July I was watching a group of four-year-olds on a slide situated outside their playroom window, to which they had ready access at all times except when it was raining. Some walked up the steps in ones, others took two or three at a time; some went down the slide head first, others feet first; some on their tummies, others on their backs; some went on their feet, walking, jumping or running down. These children were thoroughly at home. This was a space they knew, and they could relate themselves to it in various ways. One little newcomer stood watching the children for some time and later in the morning, when the slide was unoccupied, I noticed her approaching it cautiously, rest one hand on the rail and place one foot on the lowest step. Evidently this was as far as she felt she could go in conquest of this new space because presently she wandered off to watch some other children at play.

Swings, see-saws and climbing frames all provide spatial experiences; each with certain relations particular to themselves by virtue of the pattern of their construction. Bodily experience on them yields movement information which will in time be used in making further experiments, and which will gradually become more consciously defined. To go as high as possible, or as strongly as possible, may be a conscious purpose and the child's actions show that he now knows how to achieve his goal. Another space has been conquered.

In this same playground on the same morning there was a large tea-chest lying on its side. One little girl crawled inside and remained there rolled up like some little hibernating animal for nearly an hour. Another little girl brought along a smaller box and placed it beside the chest. She used this to help herself on to the top of the chest, and then peering through a crack in the wood she discovered the little girl inside. Presently she crawled along the top and peered over the open edge and got another view inside. What she was thinking about, I do not know, but something about the situation intrigued her for she spent a long time on the top of the box, sometimes looking through the crack and sometimes over the edge, completely absorbed in the experience she was having. Perhaps it was the experience of upside-down-ness which intrigued her.

A number of children from the same Nursery group were playing with some clay. Two of them were cutting little square pieces off a large lump and another was making patterns on a flattened piece in front of him. Timothy had collected most of the rest of the clay and made a large ball. This he held up high in both hands and brought it down again on the table with a bang. He picked it up again, appearing to notice the flattened under surface, and lifting it high let it fall again. He must have repeated this operation six or eight times and each time the lump became more flattened and increasingly difficult to manipulate because of the changing shape and thinness. This was all done in an absorbed silence so that there was no clue as to what he was thinking about, but it looked as if he was appreciating the change in shape he was effecting by his operations. At this moment another little girl approached the table, and seeing that there was no clay for her, she took up a cutter and saying firmly, 'You don't want all that,' she proceeded to cut the pancake in half and carry off her share to another part of the table. Timothy watched all this quite passively and then was overheard muttering to himself, 'But I was using *all* of it.' After a while he went round the table picking up all the little bits he could find, and resumed operations as before.

Building shapes with sand, putting sand into buckets and noticing the shape which comes out, watching water falling through space, building with bricks of different sizes and shapes, cutting shapes in dough, threading string through holes in beads, playing with jig-saws and fitting toys, blowing bubbles, dressing and undressing dolls, washing them and putting them to bed, having tea-parties and tidying the Wendy house can all be thought of as spatial education, and terms like inside, outside, on top of, underneath, down, up, a lot, a little, flat, round, square, holes, tunnels, hills, slopes, etc., may acquire their first meanings in terms of play experiences of this kind.

The school building and playground present a very different kind of space from that at home and many little children find its size and complexity a little frightening as well as bewildering at first. There is one's coat to find in a place where there are a lot of other coats, the way to the lavatory has to be learned and the special places where play-things are kept. There is the way to school, the way into the classroom and the way out to the playground, the way to the big hall and to the place where hands are washed. All this requires spatial adjustment of a considerable order, and it is little wonder that many little children look and feel a bit lost for the first few weeks. Equally though one might remark on how rapidly many children learn the geography of the school building. There are many little ways in which a teacher can help new children in this, e.g. by placing distinctive symbols which make it easier to identify one's own peg in the cloakroom and one's own towel in the washroom, by pictures on the doors of cupboards which tell one what is inside, etc.

In the same Nursery described earlier I watched the children laying the tables for dinner. Again, because this was in July, and many of them had shared in the Nursery life for nearly a year, some of them knew the pattern required quite well: two spoons and a fork arranged in a particular way on each side of each table, four mugs to be placed in the middle of each table, also a vase of flowers, and finally a chair to be tucked in under each side of the table. For these children the only thing which still presented difficulties was putting on the tablecloths. They just wouldn't go straight! One little boy was obviously lost among this competent party of helpers. He moved around with a handful of spoons, placed one on a corner of a table, moved to another table and scattered the rest around. His plight was soon noticed and evidently appreciated by one of the little girls, who collected the spoons up again and taking him by the hand showed him where they had to go.

All children enjoy moving about in space in different ways: running round in circles, jumping up into the air, spinning round on their own axes, walking backwards, crawling, sliding, seeing how far apart they can take their legs, how high their arms can stretch, how small and how big they can make themselves. In modern movement education this delight in movement of all kinds is used to train the children in awareness of their body-space and of their body in space, and in this sense may be thought of as a contribution to the kind of education we are thinking about in this chapter, as the following quotations from The Ministry of Education's publication, *Moving and Growing*, will serve to show.

> While young children certainly make use of movement for utilitarian purposes, they often indulge in it for its own sake, exploiting it in many ways. When they run they may get somewhere, their leap

may carry them over the wall or up to the top of the tree, but from a purely practical point of view these feats are usually unnecessary—it is the sheer act of running, leaping, or climbing, that they love.

A wide space will often set them off, skipping, hopping, running and shouting, whirling their arms and performing all sorts of—to the adults—curious antics. One may be seen walking rather exactly along a curving line, muttering to himself; another appears to find satisfaction in jumping up and down, making his impact with the ground as terrific as possible; and another indulges in a swaying, rolling motion. . . . In climbing a tree a child becomes more versatile as he finds his way round an awkward angle, and along a branch where he must wriggle full-length. He begins to know more about trees and about himself, and he gains new realization of height, width, depth, weight and resilience. . .

Children explore new ways of using their body by discovering how to overcome some physical obstacle, or by inventing stunts which involve curious forms of running, leaping, and swinging, or hanging upside down. They gain more control by repetition which is part, for instance, of bouncing a ball, or of balancing along the top of a wall. They find further use and control of their body in expressing their imaginative ideas when erecting, for example, a ship out of a pile of timber and manoeuvring it into port; or when engaged in dance or drama. . .

In dance, movement is enjoyed for its own sake—for its quality, shape and pattern. A dancer who leaps, glides, or stamps, enjoys the essence of leaping, gliding, or stamping; his is a different attitude from that of the gymnast who in leaping vaults over an obstacle, of the housewife who glides her iron over a cloth, or of the child who stamps his foot in fury.

While music lies in the field of sound, *movement lies in the field of space*; through dance, which is not tied down to utilitarian purposes, it is possible to explore and exploit all qualities of movement in space.[1]

But children also delight in things which are moving. Many a mother has learned to place her baby's pram in a position where it is possible for him to watch the movement of branches and leaves on the trees above him, knowing that thus occupied he will remain happy for longer than if he has nothing moving to watch; and many a little child has been distracted from distress by the dangling of Daddy's watch. Later the movements of ripples on water, the food-mixer, the door handle, the lavatory chain, the rolling-pin, soap bubbles, balloons, hoops, jets of water, a spinning top, a bouncing ball, the see-saw and the swing become things that are of interest to watch because of the patterns they are making in space.

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[1] *Moving and Growing*. Prepared by the Ministry of Education and the Central Office of Information.

Many a young man extends this interest still further before he reaches the Junior School, and is prepared to watch the movements of cranes, cement-mixers, road-drills, sewing machines, and pulleys for as long as the adult is prepared to wait. I remember being very impressed the first time I read Susan Isaacs'[1] account of the interest in the electric light pulleys expressed by the Malting House children. In response to their questions about how they worked she did not give an elaborate verbal explanation but instead arrived on the next day with some more which the children could handle and discover for themselves the answers to their questions. No better example could be given of the rôle of the teacher in this kind of education.

In a recently published pamphlet Nathan Isaacs gives an indication of the scope of the interests, inquiries and explorations of the young children in Susan Isaacs' school in Cambridge, which would repay careful study on the part of any teacher who is concerned to extend the range of the educative opportunities she offers to children. Commenting on his summary Nathan Isaacs says:

> I feel sure that many of these interests and inquiries could be paralleled not only from other modern Nursery Schools, but also from the experience of most sympathetic parents with their own children. . . . They are not usually kept up for long; either the child's attention shifts or fact-seeking curiosity turns into make-believe play or carefree fantasying. All we can say is that the varied interests and questions are there, often recur, and can frequently be led on, above all in the

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[1] Isaacs, S., *Intellectual Growth in Young Children.*

atmosphere of a group of eager, live-minded children stimulating one another, towards deeper penetration into the inter-connections and larger schemes of things. And under these conditions the beginnings of both the spirit and attitude of scientific inquiry can be very clearly discerned. The question is what we can make of them.[1]

The study undertaken by Celeste and John Navarro of the development of scientific concepts in their own son between the ages of three and six years is of particular interest in this context. Concerning the general findings of the study, Navarro writes:

> The progress evidenced in the child's activities seemed to be one which relied upon *interpretation of the events in his environment*. This interpretation was *based upon the child's own observations and depended heavily upon logically thought out relationships within the child's experience*. The process was one which seemed to facilitate growth, since the logical relationships were continually revised as further observation expanded the matrix of experiences. However, growth was not evidenced merely as an additive process in which new experiences were accumulated; the growth was also a matter of integration in which new relationships were drawn among both old and new experiences.[2]

He offers many fascinating incidents from their records to illustrate the way in which concepts were continually being enlarged and revised:

> As information accumulated (e.g. about materials which melt) it provided the child with an expectation in relation to an event or concern. Then, in future encounters with the items of the concern, the expectation served the child as an hypothesis. Contrasts were made by the child between information derived from his experiences and the expectations which had been formulated on the basis of past information. *The testing of his expectation or hypothesis—on the basis of further information derived through more extensive contacts with the items of concern during his activities—was the prime instigator in the conceptual growth and revision.* (Italics are mine.)

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[1] Isaacs, N., *Early Scientific Trends in Children.*

[2] Navarro, J. G., *The Development of Scientific Concepts in the Young Child.*

Navarro's records offer many examples which illustrate the fact that in young children the initial stage in the development of many concepts depends upon a self-reference. Thus both animate and inanimate objects are endowed with feelings similar to their own, and the questions asked at this stage are a logical outcome of their tendency to relate things themselves and of the self-image as seen by them in other people and things. For example, Navarro's son thought that cows should have a house to live in, 'You know, Mom, it gets dark and it rains and they have no place to sleep. They might be scared'.

Nevertheless the projective aspects of Navarro's son's self-reference were 'accompanied by a somewhat simultaneous item-by-item differentiation of his own physical characteristics and those of animals'. For example he showed intense interest in the modes of locomotion of such creatures as four-legged animals, worms and birds, as distinct from his own.

The animistic and anthromorphic interpretation of events have been noted by many writers, including Susan Isaacs and Piaget. They give rise to questions that are appropriate only in respect of human actions. But at the same time others occur which are seeking for reasons and causes for behaviour observed in objects themselves. In time the latter type replace the former entirely, but teachers of young children will meet both and need to understand the reason why a question is being asked if she is to satisfy the questioner.

Nathan Isaacs discusses another type of question which does not seek merely information but explanation:

> It arises not from a simple blank of ignorance which the questioner wants to have filled in, but to an all-too-positive state of confusion and helplessness which he needs to have removed. . . . Puzzlement is a state of disorientation and at-a-lossness. To understand it we must understand the previous state of orientation and implicit confidence. The simple key is that *even children down to three to four years have already built up within their minds a quite far-reaching working model of their world,* a sort of proto-science of the kinds and constancies of things and the regular rules of their behaviour and relations. This guides them in

> their ways about their world, enables them to recognize objects and happenings they meet, and thus permits them to foresee what to expect and how to respond and plan and set. But periodically something goes wrong; their proto-science is, after all, still very imperfect and in particular very untested. They are too apt suddenly to find themselves helpless and hurt because some assumption or belief, implicit or explicit, has collapsed upon them. And we see the effect of this in their sensitiveness to even small jars—to any observation which clashes with their existing scheme of thought, or for which this has badly failed to prepare them. They then pick up the why question . . . as the recognized way of expressing their at-a-loss state and invoking the adult's help in coping with it.[1] (Italics are mine.)

This serves to emphasize the necessity for adults to try to get at the activation behind children's questions. We can let them down badly if we don't do this, either by dismissing as trivial a question which is highly important to the child or by indulging in an elaborate explantion of an event when a simple answer is all that is needed. Records of children's activities of the kind we have been considering also serve to highlight other needs of children. Always in these studies there is evidence of a permissive relationship between children and adults which allows them to feel free to explore their environment, to share their interests with the adult and to ask for information and explanation when they feel in need of it. Always too there is evidence of a rich environment to explore and an adult who understands the kinds of things which provoke curiosity and inquiry in the young, and who goes out of her way to provide them.

Some teachers I know have sought to provide for this kind of extension of experience by providing a 'Discovery Box', or having a table or shelf reserved for such things as the insides of old clocks, clockwork toys, magnets, a magnifying glass, pieces of mirror, sea-shells, prisms, etc., and a recent pamphlet entitled *Scientific Interests in the Primary School*,[2] includes many suggestions for the teacher who is interested in building up a collection of equipment and books to help children's investigations. As the writers suggest, experiences of this kind might be thought of as providing for the beginnings of scientific inquiry. They are also a means of discovering new spatial relationships which is our special concern here.

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[1] Isaacs, N., *Early Scientific Trends in Children.*

[2] Allen, G., et al., *Scientific Interests in the Primary School.*

The children shown in Plate VIII are nearly seven. They have gone a long way in their exploration of spatial relationships and with the help of a teacher who understands the kind of situations which provoke their curiosity they have explored the principles on which levers, pulleys, ramps and magnets, work. When I happened to visit the school one day they showed off their models with great pride and were able to explain to their classmates how they worked. It was clear to me that both minds and bodies had been busy on this project and that the experience was deeply satisfying. Before I left I asked whether anyone could write me something about what they had been doing, and the following is one of the communications I received.

> I am going to tell you all about *rollers*, *wheels*, *ramps* and *levers*. If a man wanted to get a boat off the sand into the sea it would stick and it would be heavy to pull and push so he puts rollers underneath. At school if we want to move a heavy box we tilt it one way and put a roller under it: then we put another roller under it and moved it. At school we have a heavy brick box which would be hard to push if we had not got wheels on it. Wheels make work easier. If we put a plank of wood against a wall we can push a wheelbarrow up to the wall on the plank. The plank is a ramp. Animals get out and into vans on a ramp. At home I have a toy horse-box.
>
> I helped my Dad open a box and take the nails out. He pushed a bar between the lid and the box and forced the lid up. He put the claw of the hammer underneath the nail and pulled it up. The bar and the hammer are levers.

These children had made use of a number of booklets in the *How it Works* series published by Blackwell. They had been able to digest a good deal of the information contained in the text and the illustrations in these little books seemed well-suited to their level of ability. A number of firms are now publishing books of this kind and a list of some which might find place in the book corner is given at the end of this chapter. It would seem to me, however, books of this kind will have only a limited usefulness unless, as in this case, the children are given materials with which they can experiment for themselves and have a teacher ready to share and extend their interests.

One might say that these children were embarking on a study of mechanics. Although they are as yet unable to express the principles on which levers, pulleys, magnets, etc., work they are discovering through first-hand experience, the jobs which tools of various designs can do and some of the ways in which man has learned to make his own work easier. These children's observations and questions were at a very practical level but it seems likely that what they had learnt will remain with them and provide some of the basic experience to be drawn upon at a later stage in their education. Words like pressure, force, and attraction, had been added to their vocabulary even if they could not as yet define them in very precise terms.

The machines used by builders, farmers and workers on the road fascinate most children, and whether it be a pneumatic drill, a mechanical digger, a combine-harvester, a crane, or a milking machine, they soon begin to ask questions about how they work, and for names of parts which they have observed often with attention to minute detail.

Domestic machines such as refrigerators, vacuum cleaners, washing machines, sewing machines, and the lawn mower are also sources of keen interest as are the tools mother uses when cooking, such as the grater, beater, mincer and icing equipment. Having a go oneself is possible here and gives the child an opportunity to find out some of the answers for himself. This is what had happened to the seven-year-old girl shown using a sewing machine in Plate VI.

Most boys and a few girls become intensely interested in motor cars, trains, aeroplanes, space-rockets and ships while still in the Infant School. These form the subject of many of their pictures, models and free writing which often testify to closely observed experience and an interest in detailed connections.

The following are pieces of writing by seven-year-old boys and reflect the enthusiasm which was also evident in their conversation which was full of technical terms.

1. I have been to Birmingham airport and I have seen lots of aeroplanes and there was a little blue one that kept going off and coming back and there was a man with bats who showed the aeroplane where to go by waving the bats about and I had tea at the airport and they have got a cafe at the airport and it is like going up to a lighthouse when you go up to the cafe and I saw one take off and I saw them putting petrol in to an aeroplane and I saw them pushing one into one of the hangers and the hangers are garages for aeroplanes. When I was at Birmingham airport I saw them mending the runway and they were mending it with slabs and cement and they had about five hangers and they had about 20 aeroplanes and they had lots of big ones and lots of little ones and the little ones were all different colours. In the cafe it has got lots of windows in it and I looked out of the windows and I saw the aeroplanes coming in and taking off.

2. There are some new trains called deisels and I have been on a deisel and they have got very big windows in them and they have only got some glass between the driver and the passenger. It has only three coaches and they are green and they have got two chimneys and they have got them on the first and last coaches. It has got the deisel motors under the train just by the side of the train and when it goes fast it makes a lot of noise. When I went to L—— on the deisel I came home on it as well and when it came into H—— station it stopped and I walked to the front to go over the bridge and the train was still in the station and I saw the driver and the deisel motors and I think the deisel motors run all the way down the train.

3. I am interested in deisels. I have been on a deisel when it goes about seventy miles an hour it rocks about. When I went to Leicester on it it rocked so much that I fell down in the deisel. When we where coming back to the station the deisel engine broke down and we had to stay on the deisel Half an hour then when the deisel went through

the tunel it met a steam train on the same line and the deisel went and as soon as it was in the station it broke down again and it had gone on the wrong line and we had to stay in the station another hour and at last it went and we got home at half past eight.

Deisels are very nice to ride on. The exsast is on the top of the deisel. I nowtist it went I got of the deisel and I went up on the bridge and looked over the side and saw it. It came puffing out at me and it puffed right at my face. Deisel have no smoke. But smoke comes out of the exhaust When it goes out of the station more smoke comes out of the exhaust and when I went on it I saw the exsaast it was black and grey steam. trains are as nice as deisel. I have been on an express in London.

4. Ships are in the Harbar. Some times the big ships sail throgh the British cannel. I saw the queen Mary at le-on-solet and I saw a tanker and tug Boat. the tanker looked ever so little to the queen mary I did not see the queen ellizeberth Because we were too late it had just gone out of sight. Ships are very nice to ride on and I have been on a ship. It had a name. Its name was Mayflower. it was not a big ship it was a little boat. I hung my boat out and it sailed in the sea. I like to ride in ships they are very nice to ride on. I am going to have a trip on one and have a trip back as well. I have been on a paddle steamer and a little boat a tanker and a liner. Across the sea there is an island and there is a Harbar across the sea were very big ships there. When I looked thought my daddys bnockylers I saw it a I saw a tanker just going in.

Ships sail all round the would and they sometimes sink but not the queen mary not even in the English channel. A tanker can sink in the English channel.

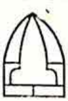
I like to take pictures of ship I took a picture of the queen mary and some other ships. One ship was a great big tanker and it was neally as tall as the cliff and as long as the cliff.

It is fairly obvious, I think, that these children have parents who are sympathetic to their interests and perhaps fathers who share them. At school there are many books available to them through which they can gain additional information and naturally these are very popular. The possibilities for extending understanding of spatial relationships through this group of interests are too numerous to discuss here. As an example one might consider some which children will come across as they watch trains, build model railways, play with train sets, look at pictures and read about trains, and reproduce their experience in drawings and writing. For example, there are the different angles made by signals; parallel railway lines and what seems to happen to them in the distance; the shapes of tunnels particularly when one is sitting at the front of a diesel train and can see the little hole at the farther end gradually increasing in size as one approaches; the action of train wheels, points and switches on railway lines, level-crossing gates; and the shapes of different engines, which enables one to classify them as express, diesel, electric trains, etc. One little four-year-old boy I know explained that the little hole at the end of the tunnel 'isn't really little, you know. It looks like that because it's a long way away.' Six-year-old John writes:

> I go train-spotting and I can see trains from my bedroom window. I have got a train book at home and I live at —— and it is near H—— station and 9 er's and 6 er's come through H—— and the Royal Scott and the Royal Scotts grey come through Birmingham. And the scotts no. is 46001 and the scotts greys number is 46002 and I have been on three deisels. And the Flying Scottsman is in London, and its number is 60136. I saw a train this morning and its number was 40207 and it was a tanker and I saw 75058 last night. Diesels are very fast they can go at 70 miles an hour or moor some old expresses come through H—— on the Birmingham rail. The wind cheater does not come through H—— very often now.

Before they leave the Infant School many children will be ready to put their spatial understanding to constructive use. This is clearly seen in the time and care devoted to elaborate building operations if there is space available and an interesting variety of bricks. It is also seen in the increasing elaborateness of things made with wood if adequate tools and suitable wood are provided. I remember watching a model of a fair growing over a couple of weeks during which time a group of six boys devoted most of their free time to this project. The final result included a roundabout, a helter-skelter, swings, a toy railway, and four caravans complete with furniture, one of which was wired for electric lighting and TV. It is interesting to consider the range of spatial problems which had to be solved in this operation.

Some of these older children enjoy following instructions for building models, such as those provided with sets of meccano. These could be extended by the teacher to instructions for the woodworkers, the following of directions on a simple map, etc.

Reggie had only just arrived in the Infant School from a Nursery class when I joined him while he was playing with an assortment of bricks one afternoon. He was trying to fit a brick into the hole under an archway he had built and when it wouldn't go he explained to me that it was because it was 'too big and not curved enough'. It was not long before he had found the shape he was looking for. Then he told me that he was building a drawbridge and it transpired that he had recently been to see the Tower Bridge in London and 'the castle where the queen lives'. He was not able to tell me about the opening of the bridge but instead pointed to part of his construction shaped thus  and explained that 'the castle looks like this'.

This ability to notice and remember relationships of size and shape is not unusual in children of this age as any teacher who finds time to talk to her children while they are playing will know. This little boy had not only had an experience outside school which had been truly educative, but because of the variety of shapes available for constructive activities he was able to relive the situation and to consolidate the gains he had made.

The two drawings in Fig. 17 were done by a little boy whose chronological age was 6¾ but who was so backward in respect of language and social skills that he was being kept in the Reception Class. He spent the greater part of every day absorbed in making drawings of this kind, and though paints and other materials were available he never used them, nor

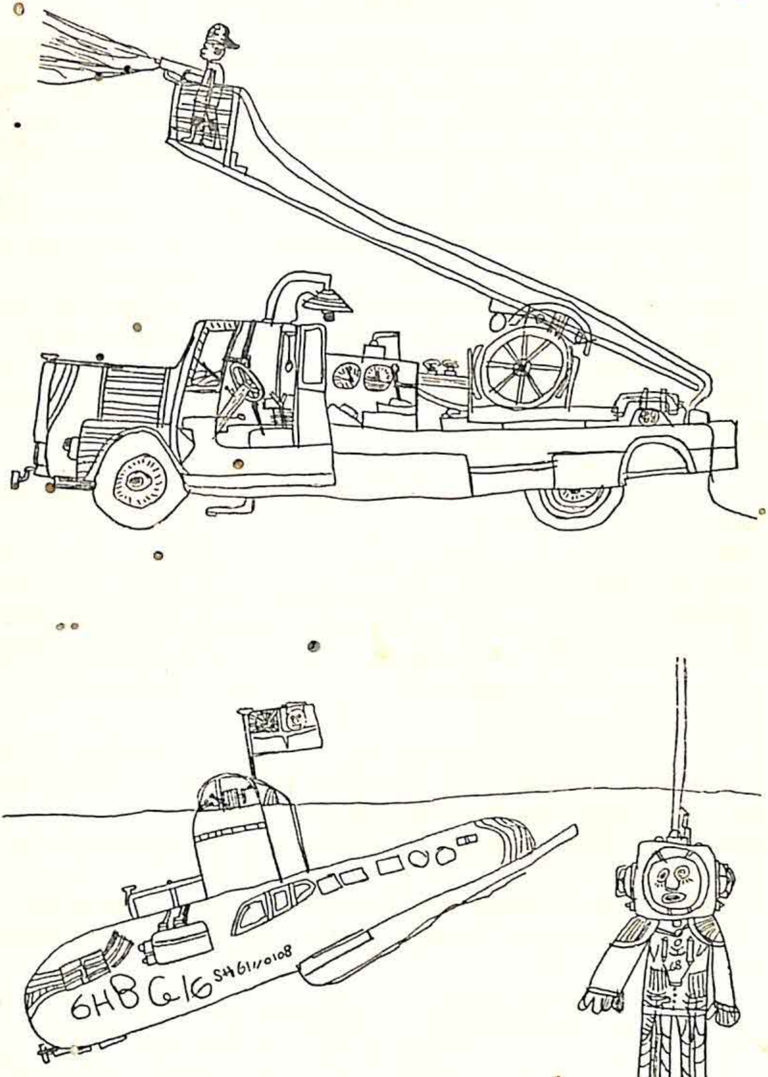


Fig. 17. DRAWINGS BY A SIX-YEAR-OLD BOY

could he be persuaded to say anything much about his drawings. Eventually his lack of progress gave cause for such concern that he was examined by the educational psychologist who assessed his I.Q. as 62 on the result of a Stanford-Binet test. He was eventually placed in a residential school for backward children where he soon began to make progress in social adjustments, extending his special gifts to modelling and to acting where he showed unexpected gifts for characterization. Progress in language skills was slow, however, and it remained difficult to determine the extent of his mental ability. Clearly he had considerable ability to make sense of the visual world he was experiencing, although when judged by attainment in the usual school subjects his capacity seemed to be very limited.

In one school I know the children are learning a good deal about shapes through picture-making with wools, cottons and odd materials. This is a popular activity particularly with the girls who enjoy inventing new stitches for decorating their pictures and show particular care in reproducing shapes and sizes when cutting out houses, people and other objects. The Fair in Plate VII is an example of their work, and the models were made by children in the same school. The niceties of detail express appreciation of the spatial properties of a bubble car, a wood-truck and the various amusements to be found at the Fair. They suggest an awareness of the relation between form and function, which is the result of closely observed experience.

At Christmas-time another opportunity for shape-making comes into its own. Making paper chains, mobiles, plate doyleys, etc., call for invention and skill and can be a very satisfying occupation if the teacher is interested enough to depart from the traditional link paper chains which decorate so many Infant classrooms at this time of year.

It is a characteristic of early childhood that interests are often fleeting but the quality of the experience can vary very much and this often determines the length of time an interest is sustained as well as its depth and extent. It cannot be too strongly emphasized how important the adult is in all this. If she is prepared to share interests and has a contribution to make from her greater knowledge and experience then the child's gains are infinitely greater. In all the experiences discussed in this chapter there are possibilities for the abstraction of spatial concepts but many of them will not be reached without the help of an adult even if the child is capable of dealing with them. This means that the teacher who wishes to exploit these experiences to the full must be prepared to extend, if necessary, her understanding of the principles of mechanics, the laws of action, the mathematics of space, etc. She must know enough to be able to appreciate the possibilities for learning in the situations which arise as well as knowing how much information, and in what form, can be digested by children. One very important feature of this communication is that she is giving the children a language in which to handle the ideas which are forming in their minds, opportunities to clarify their thinking and starting points for new lines of inquiry.

In the words of the Mathematical Association's Report:

> It is evident that there are unlimited opportunities for children to have experience of the spatial aspects of mathematics. As in number, their need at the Primary Stage is for a wealth of such experience; from this experience, with the aid of a perceptive teacher, they can increase their intuitive awareness of the behaviour of objects in space and gradually build up a body of ordered knowledge.
>
> Throughout the Primary years a child is an active being, learning chiefly through what he does. The emphasis in this chapter has therefore been on the children's own discoveries and experiments. Yet learning also takes place through a child's efforts to express himself. Speech and drawing are his main forms of expression. Conversation with his teacher and with his fellows will be an important means of clarifying his own thinking, of starting new lines of thought and of acquiring the words in which he can clothe his ideas. As the skill in writing emerges during the period he will wish to write about his experiences. We may expect that children will take pleasure in

recording what they have done in their experiments with number and space; this they can do more readily since drawings and figures will play a large part in the records, thus reducing the still laborious use of words. Such records, gradually becoming more systematic, help to give precision and permanence to what is learnt, both sharpening perception and strengthening memory.[1]

The goal towards which we are working is the development of an abstract idea of space, and though this achievement is still some way ahead in the future, it is important for the teacher of little children to understand the connection between the kind of things discussed in this chapter and mathematical space which will be classified in the concepts of the topologist and geometer.

What the young child is doing forming patterns of thought through his handling of physical space, and gradually by generalizing from his experiences, abstracting one spatial concept after another. The story of growth is similar to the one which has already been described with regard to numbers, and the concept of roundness might be taken to illustrate it. Many of the objects handled by little children are rounded—his ball, the finger of his mother, the thumb he sucks, the cup he drinks from. Their shape compels him to handle them in a certain way, and a way which is different from the way he handles objects of other shapes. Because of this he will in time abstract the particular quality of roundness which they all have in common, and will need a word for it. Maybe hearing the word used in different contexts will itself be the experience which helps him to isolate this particular characteristic from others in the experience. This is the kind of development which is going on through the child's Infant school years and with the help of a teacher who appreciates its significance a firm foundation for later work can be laid.

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[1] Mathematical Association, *The Teaching of Mathematics in the Primary School.*

SUGGESTIONS FOR INFANT SCHOOL EQUIPMENT

Large play apparatus which gives children experience of moving about in spaces of different patterns:—climbing frames, swings, see-saws, slides, an old car or cart, trees, large logs and tree trunks, a piece of cement tunnelling large enough to crawl through, wooden crates large enough to climb inside, a tent.

Playthings with varied spatial properties:—balls, bats, hoops, skipping ropes, trucks, tricycles, scooters, pedal cars, perambulators, a rocking horse.

Building materials of various sizes and shapes:—large and small bricks, large empty wooden chests, cardboard cartons, planks and pieces of wood, oddments of all sorts.

For discovering pattern, shapes and properties of sand and water:—vessels and buckets of various shapes and sizes, sieves, strainers, funnels, tubing, spades, trowels, spoons, rakes. Special equipment for making soap bubbles.

A Wendy House which provides the spatial problems of a confined space:—table, chairs, equipment for meals, cupboard, dresser, toy stove, saucepan and kettle, bed, dolls, doll's pram, wash basin, wringer, clothes horse, toy iron and ironing board, toy telephone with live connection to another somewhere nearby, shopping bags.

For making shapes with clay and dough:—tin or plastic shapes and cutters, rolling pin, modelling tools, things for additional decorations, e.g. feathers, shells, coloured stones.

For attention to finer distinctions in shapes:—jig-saws, mosaic blocks, fitting toys, hammer toys, beads and thread, model railway equipment, model cars and aeroplanes, model farm-buildings and animals, doll's house, picture books.

Box of odd materials useful in model-making:—string, wire, pipe-cleaners, dowling rods, paper clips and fasteners, needles and thread, scissors, paste and fish glue, brushes, pieces of cloth, fur, lace, feathers, silver and transparent coloured papers, tin foil, coiled wire, elastic, etc.

For modelling in wood:—a stout woodwork bench, workable wood and adequate tools, e.g. assorted sizes of hammers, saws, screwdrivers, nails, tacks and screws, bench hooks and vices; pincers, wire-cutters, drill, chisel, sandpaper, steel rulers.

Equipment and materials for cooking:—weighing scales, basins, cups, jugs, spoons, mixer, shifter, grater, rolling pin and board, cake and bun tins, tin and plastic shapes and cutters.

For making collections of odd and beautiful shapes:—shells, fossils, pebbles and pieces of rock; bones of animals, birds and fish; feathers; twisted roots and twigs, leaves, pieces of bark; mosses; lichens and sea-weeds.

Equipment for displaying in attractive ways:—pieces of mirror and coloured glass and paper, china and glass dishes, glass tanks and bowls, compartmented boxes and labels, and, of course, a magnifying glass!

Items for a Discovery Corner—the possibilities here are endless and the following are offered as examples:—Magnets—bar and horseshoe, steel knitting needles to magnetize, nails and other things to pick up. Mirrors—convex, triple and horseshoe. Locks, padlocks, hinges; door knobs. Springs—both coil and spiral variety to be found in disused clocks and watches, clockwork toys, spring balances, insides of old chair seats. Syringes and sprayers such as pen fillers, scent sprays, water pistols, oil cans, fountain pens. Pulleys in toy cranes, levers in train sets, electric torch, bicycle bell, bicycle pump. Mechanical puzzles in wood and metal.

Places to go to and things to look at with parents and teachers.

1. *In and from the Sky*: cloud shapes, sunshine effects, rainbows, eclipses, moon shapes, star patterns, snow falling and lying, snow crystals, ice, frost and rain. How things look in foggy weather. Thunder and lightning. What the wind does to trees, clouds, weather vanes, kites, balloons, and people'sh air. Aeroplanes, helicopters, jets, parachutes.
2. *At the Railway Station*: train engines of various kinds—wheel patterns, methods of locomotion, etc. Railway lines, points and levers, signals. Luggage carriers, slot machines, weighing machines, turnstile. Bridges and level crossings. Smoke.
3. *At the Seaside*: ships distinguishable from each other by shape and

size, e.g. liners, warships, ferry-boats, aircraft carriers, paddle steamers, rowing boats, motor-boats, canoes.

Parts with special functions: funnels, engines, paddles, rudder, anchor, etc. Lighthouses, lightships, lobster pots, fishing nets, piers, breakwaters, seashells, seaweed, sea creatures, and, of course, sand and water.

4. *In the Roadway*: cars, bicycles, buses, etc., to be identified by shape. Traffic signals, belisha crossings, road signs, police directing traffic, street lighting, pillar boxes. Road repairers at work: tractors steam rollers, pneumatic drills, cement-mixers, tar-sprayers and heating rakes, mechanical diggers, pick-axes, etc. Pipes exposed by holes in the road.
5. *At a farm*: milking machines, ploughs, reapers, combine-harvesters, mechanical shearers, etc.
6. *Watching builders at work*: digging foundations, mixing cement, laying bricks, putting in windows and doors, laying tiles, using spirit levels and plumb lines, carrying brick hogs, etc.

   Patterns of scaffolding, pipes for different purposes, gutterings, and drain pipes, water taps, electricity and gas fitments.
7. *In the School Kitchen*: ovens, refrigerator, washing-machine, kitchen airer for cloths, vegetable peelers, mixers, beaters, graters, mincers, etc.

   Watching Cook at work: measuring, mixing, kneading, rolling, cutting, icing a cake, etc.
8. *In the School Garden*: tree shapes, patterns in flower beds, tools for different jobs: spades, hoe, rake, trowel, fork. Lawn mower, shears. Supports for delicate plants. Sun shadows, rain puddles.
9. *In a house*: vacuum cleaners, washing machines, electric light switches, gas taps, water taps and water cisterns, electric fires, radiators, etc.

All these offer experiences of many kinds but they are presented here for the opportunities they offer for extending spatial experience, and for coming to an understanding of the relation between form and function.

USEFUL INFORMATION BOOKS FOR CHILDREN

The Observer Pocket Series. Warne.

Aircraft; Ships; Railway Locos; Weather; Automobiles, etc.

The Shown Series. Nelson.
The Starry Heavens; The Seashore.
All About Books. Publicity Products.
Ships and Boats; Mountains and Volcanos; Stars.
How it Works Series. Blackwell.
Levers; Pulleys; Magnets; Wheels; Sound; Light; Heat; Water.
I-Spy Books, News Chronicle.
Ships and Harbours; Boats and Waterways; The Wheel; Cars; On a Train Journey; On the Farm; At the Seaside; In the Street; Aircraft.
Men at Work Series. Longmans.
Coal; Textiles; Farms; Houses; Ships.
Picture Pageant Series. Macmillan.
Ships; Roads; Undersea World.
Pageant Picture Books. Ward Lock.
Ships and Sailors; Motors; Trains.
Modern Marvels Series. Ward Lock.
Man; Science; Flight.
Wonder Books. Ward Lock.
Aircraft; Ships; Motors; Railways; The Army; The Navy; Science; How it's Done; Would you believe it?; Tell me why?
Puffin Picture Books. Penguin Books.
How Planes Fly; A Book of Trains; Famous Ships; Locomotives; About Railways; A Book of Ships.
Modern Transport Series. Blackie.
Ships; Locomotives; Aircraft; Motor Cars.
Wonder Colour Books. Collins.
Timothy's Book of Trains; Timothy's Book of Ships.
Colour Photo Books. Collins.
Railways; Circus Book.
Headway Readers. Evans.
Book of: The Air; The Railways; The Sea.
Our Everyday World Series. O.U.P.
Water Tap; Railway Station; Fire Engine; Harbour.
Golden Key Books. Publicity Products.
Wonders of the World.
Nursery Colour Picture Books. Sampson Low.
Tractors; Trains; Planes.
The World in Pictures. Rathbone.
Mountain and Valleys; Icebergs and Jungles.

The Dumpy Pocket Books. Sampson Low.
Air Forces; Ships; Railways; Motors; Aircraft.
Making an Keeping Series. Educ. Supply Assoc.
A Bird Table; A Box Garden.
Looking and Finding Series. Educ. Supply Assoc.
On a Railway Journey.
Wonders of the Modern World. Parrish.
Fire; Speeding into Space; Rockets and Jets; Railways under London; Machines which seem to think; I'll show you how it happens; If you could see inside; Flying the Atlantic; A Message round the World; What's New in Flying?
Junior Reference Books. Black.
Deep Sea Fishing; The Story of Aircraft.
The Map which came to Life. G. Philip.
Signpost to the Stars. G. Philip.
The Book of Experiments. L. de Vries. John Murray.
Railway and Road Guides. Maps.

USEFUL BOOKS FOR THE TEACHER

McKay, H., *Fun with Mechanics,* Oxford Univ. Press.
The Tricks of Light and Colour, Oxford Univ. Press.
Allen, et al., *Scientific Interests in the Primary School.* N.F.F. pamphlet.
Leeming, J., *The Real Book of Science Experiments,* Dobson.

CHAPTER TEN

THE ROLE OF THE TEACHER

THROUGHOUT this book certain fundamental educational principles have been implicit, but it may be useful now to discuss them more explicitly.

Firstly, it has been assumed throughout that learning only takes place when the learner is responding actively to the experience offered to him. This is true of all learning whether by adults or children; whether we are considering the learning of motor skills, the development of acceptable modes of social response, or the acquiring of new knowledge. It is equally true of unconscious as of conscious learning, though in education we are usually concerned to make sure that what is being learned is consciously recognized. From this it follows that unless the teacher succeeds in gaining the full involvement of her children in what is going on, the physical provisions she has made will be wasted. For full involvement children must find the environment exciting; one which stimulates their curiosity and desire to experiment and find out things. They must find a challenge in it, and the results of their activities deeply satisfying. But further to this they must feel at ease in the environment, free to enjoy themselves and other people, free to get involved in learning tasks.

It is easy, of course, to write statements like these, but any teacher who has read this book will know that it is not easy to ensure this for every child all the time, and it may be helpful to consider the matter more deeply, which I propose to do under two headings: relationships in education and frameworks for education.

Children are in relationships of some kind with other people from the beginning of life. These may be good in the sense that they promote healthy growth—both physical and mental—giving the child confidence in other people and therefore in himself; building up his self-esteem; encouraging a positive, outgoing attitude towards life; and resulting in a little five-year-old who arrives at school eager for the new adventure, expecting friendly attitudes from his teacher, able to co-operate with her, and to take a lively interest in his contemporaries and in his new physical environment.

But some five-year-olds who arrive in school for the first time have not been so blessed. They have not learned to trust adults and therefore have a deep distrust of themselves. Any new experience may be a source of acute anxiety, and it may be a long time before such children come to understand that their teacher is someone who cares about them, someone who can be relied upon, someone who is predictable and dependable. Furthermore, they may find it very difficult to share her attention with between forty and fifty other children. For such children the first lessons learned in schools are concerned with these matters, and until they have gone some way towards establishing a real and healthy relationship with the teacher, they are not free to gain what the other children are gaining from the outset from their contemporaries and the physical environment. Even then such children may carry a load of anxiety due to the failure of home to meet their basic emotional needs, and this may hinder their educational progress at school.

At this point I feel I must pay tribute to the Infant teachers of this country. It is a constant source of amazement to me to watch the loving care and skill with which they handle large groups of little children, and their awareness of individual needs which is often the result of deep insight and an infinite capacity for patience. Parents should be very grateful for the many dedicated women to whom they can entrust their children with confidence.

Perhaps what a child needs most is to be allowed to be himself and to find that this self is liked and appreciated. He needs to find his efforts to establish relationships reciprocated, and will only feel secure when he feels certain that a genuine two-way relationship exists.

With this as a beginning the teacher-child relationship can become educative in the deepest and the widest sense. She will be able to help him in his process of 'becoming', and with her he can learn not only how to be on the receiving end of love and attention, but how to give, and in becoming an outgoing person to discover all the satisfactions which social life in school can bring. With her support he will also discover the excitement of exploration, experiment and discovery in the physical environment, and be in a position to acquire information and skills appropriate to his age.

This aspect of the little child's educational needs is widely recognized today and many writers[1] have dealt with it more fully than can be attempted here. As an example, in the recent UNESCO report on Education and Mental Health[2] these words are found:

> Learning in the strictly educational sense will not proceed satisfactorily if the child's emotional life is disturbed; conversely, the development of personality may be fostered or checked by the way in which a child is able to respond intellectually to the demands made by school. Hence both mental health and intellectual progress depend upon the way in which the school adapts itself to the developmental level of its pupils, and, while not abandoning its demands, adjusts those demands to processes of growth and maturation.

This means that from the time a child is five years old school shares with home in influencing the trend of emotional growth and the course which personality development takes. Pressing a child on at a pace he cannot take, or withholding adequate stimulus from the able child, may have equally serious effects on personality growth.

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[1] E.g. Bowlby, J., *Child Care and the Growth of Love*. Hourd, M., *Emotional Aspects of Learning*.

[2] Wall, W. D., *Education and Mental Health*.

Most teachers are aware that the disturbed child cannot learn in the same way as his contemporaries, but there is another aspect of this problem which needs thinking about. The mode of presentation of anything to be learnt at school will affect the emotional and intellectual response of the children. Wall draws attention to the importance in teaching of distinguishing between the logical structure of a subject, process or technique, and the psychological mode in which it is taught.

> This distinction is at the root of successful pedagogic method and deserves more attention—particularly in the teaching of arithmetic—than it usually gets. . . . Almost any given notion or process or operation found in the primary school curriculum can be dealt with concretely (by actual perception or by real action or by experience) or symbolically or abstractly, and in a more or less generalized ways. These are psychological dimensions. Moreover, the context in which the latter occur will determine the amount of energy which the child will bring to their handling. If a given notion, operation or technique is seen by the child to be of immediate interest and value to him, and if it is dealt with by the teacher in a more concrete and practical way, then it can be acquired more surely and at an earlier age than if it is unrelated to his interests and presented in a symbolic, abstract or generalized way. Thus though logically the content may be the same, psychologically very different processes are involved.

This quotation serves to emphasize the fact that throughout this book we have been concerned not only with analysing what might be considered to be a proper basic education for later work in mathematics, but also with choices of approach about which right or wrong decisions may spell the difference between mental health and ill-health. While not wishing to exaggerate this, experience suggests that one of the most common causes of emotional resistance to arithmetic is due to the kind of teaching which presents concepts in the form of definitions, and processes as rules of thumb. Children have to feel and act their way into concepts, and resistance to any other way of learning might well be considered a healthy reaction on the part of the child were it not for the fact that it sets up an emotional pattern which affects the learning process as a whole.

The words right and wrong have an emotional flavour long before children meet the ticks and crosses which put their sums into these categories, and there is a sense in which arithmetic can be very satisfactory just because it is either right or wrong, and you know where you are. I have known many children, like Caroline, who gain enormous satisfaction from the number of ticks they gain every day, but I have also known other children who have become filled with despair as they come to expect an additional quota of new crosses every day.

What one would wish for every child is the excitement which follows the discovery of a mathematical truth because it is experiences of this kind which woo the child into full involvement in his mathematical education. Such discoveries are not reserved for older children, and little children very often show their excitement very openly. I think it was Susan Isaacs who told the story of a little girl in her Nursery School who, on making an important discovery about four, spent the subsequent half-hour running round the room, saying to anyone who would listen to her, 'I say, do you know—two and two make four!' Molly Brearley records a similar event: 'A small boy rushed up to his teacher and said, "Do *you* know what 39 and 17 are?" "No" said the teacher, "do you?" "56". "How did you get that?" asked the teacher. "Oh, I just *thought* it," was the reply, "I *often* think things."'[1] Many of the discoveries by children described in this book were accompanied by the obvious signs of satisfaction which are such a joy to watch.

As Wall says:

> The motive power of learning is the emotional life, and interest is one of the ways in which the emotions point outward to an analysis of the objective world. Active learning takes place when the whole

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[1] Brearley, M., *Number in the Primary School.*

emotional energy is concentrated on an interest and when intelligence is exercised in its service.

This is what we must try to secure for all children in our schools.

Turning now to what I have called the frameworks of education, I want to discuss certain other conditions which must obtain in the school environment if learning is to proceed satisfactorily. These, too, relate to the psychological needs of little children. We have already stressed the need for children to be active participants in their own education, and because of the modes of learning open to little children which were discussed in earlier chapters, this means allowing space and freedom for bodily movement as well as mental activity, and the kind of permissive atmosphere which allows the child to feel free to explore and experiment, though at the same time providing him with the kind of support and controls without which he cannot feel free.

Space is a big problem for many teachers with small overcrowded classrooms, but in many modern schools where there is direct access to the playground from the classroom this problem can be solved by having children working both in- and out-of-doors whenever possible. Some teachers make more use of corridors and cloakrooms than others, and this seems to be related to the extent to which they feel they can trust their children to be busy without constant supervision.

Freedom implies freedom to make mistakes and messes from time to time, as well as freedom to achieve successes through one's own efforts. The child who is afraid of disapproval is not free to use the environment provided as fully as he might. Freedom is sometimes identified with licence to do what one likes. This is not what is meant here. The little child is not an adult and should not be expected to display the same sense of responsibility and adequacy to deal with situations. He needs guidance and control, and knows that he needs them. This kind of framework is essential to his feeling of security, and what the teacher has to aim at is a delicate balance between freedom and control, suited to the particular group for which she is responsible. As has been said already in an earlier chapter, teachers vary very much with regard to the kind of balance they think right and proper, which suggests that this is related to a difference in personality structure which allows one teacher to be freer than another in her relationship with children.

The correct timing of educative opportunities is another aspect of the framework. Psychological research suggests that there is for every child an optimum age for learning anything, i.e. a mental age at which he will absorb it more readily than either earlier or later. In the past our most frequent mistake was to offer experiences before the children were ready, but there is enough evidence from the kind of educational casualties who arrive at the Child Guidance Clinics in need of special remedial care, to make us fully aware of the dangers of doing this, and much frustration for both children and teacher will be prevented if we heed the lesson. Many children become bored and lose interest in learning when they are offered experiences which they cannot digest, a few will develop acute anxiety states. In either case they have been deprived of the excitement which comes with the dawning of a new understanding and the achievement of a new skill.

Perhaps in these days, at any rate during the Infant School period, we are more prone to make the mistake of withholding educative opportunities for which some of our more able children are ready. This can be equally frustrating for the child and may cause such a serious withdrawal from school education that he too finds himself in a Child Guidance Clinic.

Probably most of my readers will accept the soundness of these statements but be aware at the same time of how extremely difficult it is to provide in a really satisfactory way for the range of ability and variations in experience which obtain in most groups of children of the same chronological age. This is one of the reasons for stressing the need to allow the children a good deal of freedom of choice in the activities. If the environment is suitably varied most children will choose something which is challenging and satisfying to them, though there are some in most groups of children who need to be persuaded to tackle something which really challenges their capacities. Most of the materials and activities suggested in the latter chapters of this book can be used at very different levels, and will be if the teacher is aware of her responsibility in the matter of provisions and encouragement.

Turning now to summarize some of the things which have been said in this book about number education, I wish to emphasize the following points. Firstly, that we are concerned in early work in number with the establishment of certain ways of thinking which are connected with the nature of number itself and indeed of all mathematical symbolism, i.e. its relational character. This kind of thinking is not developed through verbal instruction and rule of thumb methods of teaching. Nor is it encouraged in any way by much of the number apparatus to be found in Infant classrooms at the time of writing. We have failed all too often to ensure that information acquired in everyday experience and in play activities becomes a dynamic set of relations, because we were not ourselves thinking in these terms.

Secondly, the operational character of knowledge has been stressed. We learn by operating on things with both our bodies and our minds, but particularly in childhood the bodily operation is enormously important because very little information offered in the form of verbal ideas can be translated into knowledge at this age. In time concepts are abstracted from this kind of knowledge, but only when mental abilities are sufficiently mature, and only when bodily experience is sufficient.

Thirdly, there has been an attempt to try and clarify the distinction the concrete and the abstract in this particular context. Perhaps it is unnecessary to discuss this again here, but I should like to stress the necessity of being clear about it. Far too often the provision of certain pieces of number apparatus is justified on the grounds that this is making number concrete simply because the child has things to look at and handle. It is possible, as we have seen, to provide materials which are, in fact, concrete representations of the number system, but there is a good deal of material about which does violence to the structure of number and would be better placed on the scrap heap.

Fourthly, I have tried to analyse out the basic concepts about number and quantity which the child needs to understand before he can undertake with any degree of understanding the kind of operations which we include in Arithmetic. Laying these foundations is the primary function of the Infant School number education and all but the most able children will need all their Infant School life to gain them. Some indeed will not have them with any certainty till they have been in the Junior School for some time, and one of the recent developments which one welcomes most warmly is the growing recognition of this fact among Primary School teachers.

Fifthly, the fact has been stressed that readiness for learning to deal with numbers with understanding depends upon several factors: the coming to maturity of certain mental abilities, a basis of favourable physical and mental experiences, and on the ability of parents and teachers to recognize and use moments of readiness. It is generally agreed that readiness for dealing with number comes later than readiness for reading in most children. This is because, as Thyra Smith points out, 'Skill in reading depends to a large extent on being able to interpret marks on paper that represent words. Many words are names of familiar things and so the words are only one remove from recognition of the thing itself.'[1] In contrast, as Piaget has shown, operating with numbers calls for several mental abilities which make possible the understanding: 1. that some things are stable and invariable; 2. that these things can be grouped, and that groups have a number property of 'oneness', 'two-ness', 'three-ness'; 3. that these number properties have an order of size, a quantitative relationship to one another; and further, 4. that these quantities are represented by certain symbols agreed in our culture. As Miss Smith says:

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[1] Smith, T. *Number*.

> All this calls for a power of abstraction that is beyond the reach of young children. The necessary *conceptions should be based on real experience*. The child has many experiences before coming to school; these should be repeated, enriched and enlarged in the early days of his school life. His experiences at school will differ in degree and in the ways in which they are arranged; *they should not be of a different kind.* Number is a language which helps to express ways of thinking about things; it is a tool for use as circumstances demand and it can do both these things in the life of the school-child—if the ways in which they are done are suited to his stage of growth. (The italics are mine.)

In conclusion, if the general argument of this book is accepted then it is clear that the job of the Infant teacher is not the simple and easy one it is sometimes thought to be. It is sometimes said that Infant School children seem to do nothing but play in school these days. If this were true in the sense intended then the teachers would be very much to blame. But play, in the sense of operating on things, can and should be the means through which little children come to an understanding of the things they need to know. This places an immense responsibility upon the teacher since it is she who decides on the materials with which the children play, and it is she who by the quality of her relations with them either promotes or impedes learning. If she enjoys her life with children and caters for them with intelligence and sympathy there will be love in the learning.

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